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Heterogeneous Patience, Population Growth, and Wealth Accumulation

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Abstract

We study a neoclassical growth model with population growth and agents who are heterogeneous in their discount factors. Population growth is interpreted as the entry of new infinitely-lived agents with zero initial endowments who are not included in the economic calculus of existing agents. We prove that when capital and labor are substitutes in production and utility is isoelastic, there exists a unique stationary equilibrium in per capita terms. A stationary equilibrium can take one of two forms: a *Ramsey conjecture equilibrium*, in which only the most patient agents own the entire capital stock, or a *non-degenerate equilibrium*, in which agents other than the most patient ones also hold positive amounts of capital. We show that introducing public debt, a labor income tax or a capital subsidy shifts the economy from a Ramsey conjecture stationary equilibrium to a non-degenerate one, and analyze the resulting relationship between income and inequality.

Keywords: Economic growth; Heterogeneous discounting; Inequality; Ramsey conjecture; Overlapping generations

JEL Classification: D15, D31, D50, E21, O40

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1 Introduction

This paper studies a competitive equilibrium in an infinite-horizon model with population growth where agents are heterogeneous in their discount factors. It is well known that differences in patience generate income and wealth inequality. A defining property of equilibrium in dynamic models with heterogeneous discounting is the degenerate long-run distribution of wealth, known as the *Ramsey conjecture*.[1] In the long run, a heterogeneous society divides into two unequal classes: the most patient agents are rich and own the entire capital stock, while all other agents are poor, do not save, and consume hand-to-mouth.[2] This paper shows that when population growth is interpreted not as an increase in the size of existing dynasties (the Ramsey demographic structure) but as the entry of new agents without initial capital — the overlapping generations (OLG) demographic structure — it can prevent the most patient agents from holding the entire capital stock and lead to a non-degenerate long-run wealth distribution.

To unpack this mechanism, compare the Ramsey and OLG interpretations of population growth. Under the Ramsey demographic structure, all newly entering agents with the same discount factor belong to a single infinitely-lived dynasty. The increased population in each period equally shares the endowments from the current population, so the behavior of all agents entered in different periods can be represented by that of a single growing dynasty. The aggregate dynamics of representative dynasties that differ in their discount factors verify the Ramsey conjecture: only agents from the most patient dynasty eventually own capital.

By contrast, under the OLG demographic structure, new agents enter the economy with *zero* endowments and establish their own infinitely-lived dynasties. The current population does not include newly entering agents in its economic calculus and makes no transfers between dynasties. As a result, in each period the current population saves less than it would under the Ramsey demographic structure, which raises the interest rate and incentivizes less patient agents to accumulate capital as well. We show that the OLG demographic structure invalidates the Ramsey conjecture and alters the effects of standard policies in infinite-horizon models with heterogeneous agents.

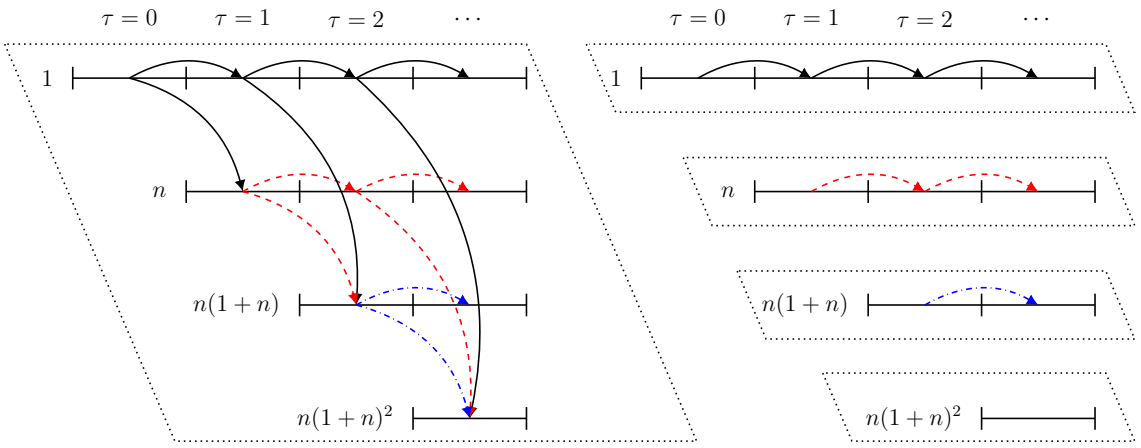
The difference between the two interpretations is illustrated in Fig. 1. In both cases, the current population sharing the same discount factor and living in period τ is represented by a vertical column which consists of infinitely-lived

[1] Originally formulated by Ramsey (1928), it was formalized by Becker (1980) and Bewley (1982). See also Becker (2006) and Mitra and Sorger (2013).

[2] In a recent contribution, Aguiar et al. (2025) show that heterogeneity in discount factors is important to match the observed share of poor hand-to-mouth consumers in U.S. data.

generations that entered between periods 0 and τ. Population grows over time at a constant rate n. Under the Ramsey demographic structure, population growth corresponds to an increase in the number of members within each generation. As a result, all generations are linked into a single infinitely-lived dynasty through intergenerational and intragenerational transfers (panel (a) on the left). Each generation transfers capital to itself and to its new members, as shown by solid arrows (generation entered in period 0), dashed arrows (generation entered in period 1) and dash-dotted arrows (generation entered in period 2).

At the same time, under the OLG demographic structure, population growth corresponds to the entry of new agents. Existing generations do not internalize the welfare of new generations, so each generation forms a separate dynasty (panel (b) on the right). In this case, capital is transferred only within each generation, leading to lower aggregate savings than under the Ramsey demographic structure, as indicated by fewer arrows. Since the economy accumulates less capital, the interest rate remains high, allowing the less patient agents to save when this mechanism is sufficiently strong.



(a) Ramsey demography (b) OLG demography

Figure 1: Intergenerational transfers under the Ramsey demographic structure (a) and their absence under the OLG demographic structure (b).

Specifically, we consider a Ramsey model with agents who are heterogeneous in their discount factors and cannot borrow against future wages, as in Becker (1980). In each period, several infinitely-lived agents enter the economy with zero initial endowments, in line with the OLG demographic structure, as in Weil (1989). We define a stationary equilibrium in terms of the per capita capital stock, and prove that, when capital and labor are substitutes in production and utility is isoelastic, there exists a unique stationary equilibrium.

We show that the stationary equilibrium is one of two types: a *Ramsey*

conjecture stationary equilibrium, where only the most patient agents own the entire capital stock, and a *non-degenerate stationary equilibrium*, where agents other than the most patient ones own positive amounts of capital. In particular, the economy admits a non-degenerate stationary equilibrium when heterogeneity in patience is small (discount factors of agents are close to each other), population growth is rapid (a large number of new heterogeneous agents with zero capital enter each period), the elasticity of intertemporal substitution is low (agents are reluctant to shift consumption across periods), or the capital income share is high (capital plays a sufficiently strong role in production).

We study the effects of fiscal policies on capital accumulation and inequality in the stationary equilibrium. We show that introducing *public debt* financed by a lump-sum tax or levying a *labor income tax* to fund public spending shifts the economy from a Ramsey-conjecture-type equilibrium to a non-degenerate-type equilibrium. Intuitively, since the current population does not include newly entering agents in its economic calculus, it disregards the tax burden imposed on future generations and saves less than it would under the Ramsey demographic structure. As a result, government deficits are not offset by agents' savings. Public spending crowds out private capital, reducing the stationary capital stock. This, in turn, raises the stationary interest rate and allows less patient agents to save, changing the type of stationary equilibrium.

In contrast, a *capital tax*, whether revenues finance government spending or are redistributed back to agents, preserves the Ramsey-conjecture-type equilibrium. Intuitively, by reducing incentives to save, a capital tax raises the threshold return on capital required for less patient agents to begin saving. A shift to a non-degenerate equilibrium occurs only under a sufficiently large *capital subsidy*, which reverses this logic and lowers the minimum required return on capital.

We analyze the impact of changing the type of stationary equilibrium on wealth inequality. Since agents differ both in age (the period in which they enter the economy) and in patience (discount factor), inequality is two-dimensional. We show that a shift to a non-degenerate equilibrium *reduces intragenerational inequality* among agents with different discount factors, as measured by the Gini index of capital distribution. Intuitively, when less patient agents also save, the capital holdings between different groups of agents are distributed more equally.

At the same time, a shift to a non-degenerate equilibrium *increases intergenerational inequality* among agents with the same discount factor who entered the economy in different periods, as measured by the 90/10 percentile ratio of wealth. Intuitively, a higher return on capital accelerates the growth of individual savings. The faster agents accumulate capital, the wider the gap

becomes between earlier and later generations. Both public debt and a labor income tax reduce the stationary capital stock, thereby raising the stationary interest rate. Although a capital subsidy increases the capital stock (unless the elasticity of intertemporal substitution is too low), the after-subsidy gross interest rate rises, leading to the same effect on intergenerational inequality.

Our work relates to several strands of the literature. First, it contributes to the analysis of competitive equilibrium in economies with heterogeneous agents. Ramsey (1928) considered a model of capital accumulation with many agents who differ in their discount factors and conjectured that, in a stationary equilibrium, the entire capital stock belongs to the most patient agent who enjoys the highest consumption level. All other, less patient agents, do not save and consume hand-to-mouth. Becker (1980) and Bewley (1982) formalized the Ramsey conjecture under different assumptions about the completeness of capital markets.

Becker (1980) studied a model with borrowing constraints where agents' savings must be non-negative. He showed that there exists a unique stationary equilibrium which verifies the Ramsey conjecture: all capital is owned by the most patient agent enjoying the highest consumption level, while all less patient agents make zero savings and consume their wages. Many properties of equilibria in the Ramsey–Becker model were further studied (see Becker, 2006; Borissov and Pakhnin, 2023, for surveys). Bewley (1982) established the existence of a general equilibrium assuming that agents can borrow against future wages and proved that any equilibrium allocation is Pareto-optimal. In the long run, in any equilibrium in the Ramsey–Bewley model, only the most patient agent has positive consumption and owns the entire capital stock along with the debt of all other, less patient agents, who consume nothing.

A natural question is when the Ramsey conjecture does not hold. Several conditions are known in the literature. Lucas and Stokey (1984) show that under a more general *recursive utility* specification instead of the common additively time-separable utility, the Ramsey model with heterogeneous agents admits a unique stationary equilibrium in which all agents have positive wealth and consumption levels (see also Epstein and Hynes, 1983). Recursive utility exhibits non-constant time preference and allows the implied discount factor to depend on consumption. When the discount factor is not constant, an agent facing a fixed rate of return on capital that is below her time-preference rate will not necessarily dissave.

Sarte (1997) shows that when agents with different discount factors face a *progressive tax*, there exists a unique stationary equilibrium in which all agents own positive amounts of capital (see also Li and Sarte, 2004; Bosi and Seegmuller, 2010). Under progressive taxation, the marginal tax rate is higher for rich agents

than for poor agents, and hence less patient agents face higher rates of return on capital, which allows them to save as well.[3]

Our contribution is to develop a Ramsey model with heterogeneous agents that incorporates an OLG demographic structure, providing a unified and flexible framework for studying long-run growth and inequality. We show that in both the Ramsey–Becker and Ramsey–Bewley settings, regardless of the completeness of capital markets, the entry of new agents without initial endowments affects the long-run capital stock and can yield a non-degenerate wealth distribution. Thus, population growth, understood as the arrival of new agents who are not included in the economic calculus of existing ones, offers an alternative mechanism that can invalidate the Ramsey conjecture.

Second, this paper contributes to the literature on OLG models without intergenerational transfers. Blanchard (1985) and Weil (1989) study economies in which new agents with infinite planning horizons enter every period without initial endowments (see also Buiter, 1988; Weil, 2008). While Blanchard (1985) assumes that agents face a constant probability of death and sets the entry rate of new agents so that net population growth is zero, Weil (1989) assumes positive and exogenous population growth. Both models analyze the dynamics of per capita capital accumulation and, in particular, show that Ricardian equivalence does not hold. Intuitively, in both cases, there are no altruistic links between different generations of identical agents, which prevents current generations from offsetting a tax cut today by saving more on behalf of future generations and blocks a standard channel of Ricardian equivalence.

Our contribution is to introduce heterogeneity in discount factors into an OLG model with infinitely-lived agents. We show that the mechanism responsible for the failure of Ricardian equivalence in OLG models (Blanchard, 1985; Weil, 1989) can circumvent the Ramsey conjecture when agents also differ in their discount factors. The key observation is that the entry of new agents with zero endowments makes the growth rate of average consumption negatively dependent on the capital stock. Therefore, the stationary capital stock is no longer pinned down by the discount factor of the most patient agents (as in the Ramsey-type models of Becker, 1980; Bewley, 1982). As a result, the stationary interest rate exceeds the discount rate of the most patient agents, and, for some parameter configurations, less patient agents also hold positive amounts of capital in the stationary equilibrium.

[3] Long-run distributions of consumption and wealth under heterogeneous discounting can also be non-degenerate under uncertainty (Becker and Zilcha, 1997; Espino, 2005), when capital holders strategically account for their market power (Sorger, 2002, 2008; Pichler and Sorger, 2009), or when agents' preferences incorporate temptation motives (Drugeon and Wigniolle, 2017).

The paper is organized as follows. Section 2 describes the model. Section 3 defines an equilibrium, proves the existence and uniqueness of a stationary equilibrium, and provides its characterization. Section 4 discusses the effect of fiscal policies, while Section 5 analyzes a trade-off between income and inequality. Section 6 studies several modifications of the baseline model. Section 7 concludes. All proofs and derivations are relegated to the Appendix.

2 The model

In this section, we describe the main building blocks of the model. For clarity, we present a version with two types of agents differing in their discount factors. A general setting is discussed in Section 6.

Time is discrete and runs from $\tau = 0$ to infinity. The economy consists of many infinitely-lived agents of two types: most patient (H) and least patient (L). In each period τ, given the current population N_τ, nN_τ new infinitely-lived agents enter the economy, so that the population in period $\tau+1$ is $N_{\tau+1} = (1+n)N_\tau$. Let π_i be the initial share of type $i = H, L$ agents in the population, with $0 < \pi_i < 1$, and $\pi_H + \pi_L = 1$. We assume that new agents of each type enter in the same proportion, so that the share of each type remains constant over time.

The economy produces a single good that is either consumed or invested. The production side is characterized by a neoclassical production function $F(K, N)$, homogeneous of degree one. The production function in intensive form, $f(k) = F(K/N, 1)$, where $k = K/N$ is the per capita capital stock, is monotonically increasing and strictly concave. Capital depreciates completely within one period. In each period τ, producers maximize profits, so that the gross interest rate $1+r_\tau$ and the wage rate w_τ are equal to the corresponding marginal products.

On the consumption side, each agent of type $i = H, L$ is characterized by the constant discount factor β_i. We assume that $1 > \beta_H > \beta_L > 0$, that is, type H agents are more patient than type L agents. Preferences of a type i agent are represented by an instantaneous utility function $u_i(c)$ that satisfies the standard assumptions: $u_i'(c) > 0$, $u_i''(c) < 0$, $\lim_{c\to 0} u_i'(c) = +\infty$.

Each type i agent entered in period τ is endowed with initial savings $s_{\tau-1}^{\tau,i}$. We assume that for $i = H, L$, $s_{-1}^{0,i} \geq 0$ are given, and $s_{\tau-1}^{\tau,i} = 0$ for $\tau \geq 1$. As in Weil (1989), only agents from period 0 may have positive endowments, while all new agents enter without initial savings. We discuss the role of this assumption below.

Each agent of type $i = H, L$ entered in period $\tau \geq 0$, given initial savings $s_{\tau-1}^{\tau,i}$,

solves the following problem:

$$\max_{\{c_t,s_t\}_{t=\tau}^{\infty}} \sum_{t=\tau}^{\infty}(\beta_i)^{t-\tau}u_i(c_t)\,, \qquad \text{s. t.} \qquad \begin{aligned} c_t + s_t &= (1+r_t)s_{t-1} + w_t, \quad t \geq \tau, \\ s_{\tau-1} &= s_{\tau-1}^{\tau,i}\,, \qquad\quad s_t \geq 0, \quad t \geq \tau\,, \end{aligned} \tag{1}$$

where the sequences of interest rates $\{r_t\}_{t=\tau}^{\infty}$ and wage rates $\{w_t\}_{t=\tau}^{\infty}$ are taken as given by the agent. As in Becker (1980), capital markets are incomplete: agents are prohibited from borrowing, and their savings must be non-negative in each period. We denote the solution to problem (1) by $\{c_t^{\tau,i}, s_t^{\tau,i}\}_{t=\tau}^{\infty}$.

Since the described model combines the features of the Becker (1980) and the Weil (1989) models, we refer to it as the *Becker–Weil model*.

We are interested in the behavior of the average consumption and capital stock belonging to the agents of the same type. The average consumption of type $i = H, L$ agents in period $\tau \geq 0$, C_τ^i, is given by the total consumption of all type i agents entered up to period τ divided by the total number of such agents:

$$\begin{aligned} C_\tau^i &= \frac{\pi_i N_0 c_\tau^{0,i} + \pi_i(N_1 - N_0)c_\tau^{1,i} + \ldots + \pi_i(N_\tau - N_{\tau-1})c_\tau^{\tau,i}}{\pi_i N_\tau} \\ &= \frac{c_\tau^{0,i} + nc_\tau^{1,i} + \ldots + n(1+n)^{\tau-1}c_\tau^{\tau,i}}{(1+n)^\tau}\,. \end{aligned} \tag{2}$$

Similarly, we define the average savings of type i agents as $S_{-1}^i = s_{-1}^{0,i}$, $S_0^i = s_0^{0,i}$, and further for each $\tau \geq 1$,

$$S_\tau^i = \frac{s_\tau^{0,i} + ns_\tau^{1,i} + n(1+n)s_\tau^{2,i} + \ldots + n(1+n)^{\tau-1}s_\tau^{\tau,i}}{(1+n)^\tau}\,. \tag{3}$$

Then the per capita consumption in the economy in period $\tau \geq 0$ is given by

$$c_\tau = \pi_H C_\tau^H + \pi_L C_\tau^L\,.$$

The per capita capital stock in period $\tau = 0$ is $k_0 = \pi_H S_{-1}^H + \pi_L S_{-1}^L$, and in period $\tau \geq 1$ it is given by

$$(1+n)k_\tau = \pi_H S_{\tau-1}^H + \pi_L S_{\tau-1}^L\,, \tag{4}$$

where we have taken into account that $s_\tau^{\tau+1,i} = 0$.

The demographic structure of the Becker–Weil model deserves several comments. Unlike typical OLG models, each agent in the Becker–Weil model is infinitely lived and makes decisions over an infinite horizon. Consequently, the number of overlapping generations grows without bound over time. A defining

property of newly entering agents is that they are not linked to existing agents through operative intergenerational transfers. Following Weil (1989), this property can be interpreted in two ways.

First, as a *primogeniture economy*: parents leave bequests only to the eldest child, while younger children initiate their own families. When parents and children share the same type, starting from two different types of parents, in each period several new infinitely-lived families of both types enter, linked through bequests to their first-born children but not to other newly entering families of their type.

Second, this property can be interpreted as an *immigrant economy*: newly entering infinitely-lived agents can be thought of as constantly arriving immigrants with zero wealth. One can think of an immigration policy which allows relocation from two different countries, patient and impatient, such that heterogeneous agents enter in the same proportion in each period.

It is instructive to compare the Becker–Weil model with a Ramsey model with heterogeneous agents, borrowing constraints, and population growth under the Ramsey demographic structure. In this latter model, referred to as the Ramsey–Becker model with population growth, the current population treats newly entering generations as members of a single dynasty (see panel (a) in Fig. 1). Hence, each type i agent entered in period τ solves problem (1) given initial savings $s_{\tau-1}^{\tau,i} = S_{\tau-1}^{i}/(1+n)$. The dynamics of average values in this model are equivalent to those of the Ramsey–Becker model with representative dynasties that differ in their discount factors. By the classical result of Becker (1980), stationary equilibrium verifies the Ramsey conjecture: only agents from the most patient dynasty own capital.

By contrast, in the Becker–Weil model, new agents enter each period with zero endowments and form new dynasties (see panel (b) in Fig. 1). As shown below, introducing the OLG demographic structure allows the Ramsey-type model to admit a stationary equilibrium with a non-degenerate wealth distribution.

3 Equilibrium characterization

In this section, we define an equilibrium in the Becker–Weil model. We prove the existence and uniqueness of a stationary equilibrium under isoelastic utility and CES production function, and characterize its properties.

A general equilibrium with production is defined in a standard way. In equilibrium, agents maximize their utilities, producers maximize their profits,

and different types of agents within each period are linked through the interest rate in the capital market, which depends on the per capita capital stock. Formally, an equilibrium in the Becker–Weil model is defined as follows.

Definition 1. *A sequence* $\{(C_\tau^{i*}, S_\tau^{i*})_{i=H,L}, k_{\tau+1}^*, r_\tau^*, w_\tau^*\}_{\tau=0}^\infty$ *is an* equilibrium in the Becker–Weil model *starting from* $(S_{-1}^{H*}, S_{-1}^{L*})$ *with* $k_0^* = \pi_H S_{-1}^{H*} + \pi_L S_{-1}^{L*}$*, if in each period* $\tau \geq 0$*:*

1. *Each agent of type* $i = H, L$ *entered in period* τ *determines individual consumption and savings* $\{c_t^{\tau,i*}, s_t^{\tau,i*}\}_{t=\tau}^\infty$ *by solving problem* (1) *at given* $\{r_t^*\}_{t=\tau}^\infty$ *and* $\{w_t^*\}_{t=\tau}^\infty$*, starting from* $s_{-1}^{0,i} = S_{-1}^{i*}$ *and* $s_{\tau-1}^{\tau,i} = 0$ *for* $\tau \geq 1$*;*

2. *The average consumption of type* i *agents,* C_τ^{i*}*, is given by* (2)*, and the average savings of type* i *agents,* S_τ^{i*}*, are given by* (3)*;*

3. *Prices are equal to marginal products:*

$$r_\tau^* \;=\; f'\left(k_\tau^*\right) - 1\,, \qquad \text{and} \qquad w_\tau^* \;=\; f\left(k_\tau^*\right) - f'\left(k_\tau^*\right) k_\tau^*\,;$$

4. *Aggregate savings are equal to the capital stock, that is, the per capita capital stock is given by* (4)*:* $(1+n)k_{\tau+1}^* = \pi_H S_\tau^{H*} + \pi_L S_\tau^{L*}$*.*

Note that in an equilibrium, a form of Walras' Law implies that in each period the total consumption demand and supply of capital for the next period equals current output. In per capita terms, for each $\tau \geq 0$, we have

$$c_\tau^* + (1+n)k_{\tau+1}^* \;=\; \pi_H\left(C_\tau^{H*} + S_\tau^{H*}\right) + \pi_L\left(C_\tau^{L*} + S_\tau^{L*}\right) \;=\; (1+r_\tau^*)k_\tau^* + w_\tau^* \;=\; f(k_\tau^*)\,.$$

It is instructive to discuss the dynamics of the average consumption in the Becker–Weil model under log-utility ($\eta = 1$). In Appendix A we show that if type i agents have positive savings in equilibrium, $S_\tau^{i*} > 0$ for all τ, then their average consumption evolves according to

$$C_{\tau+1}^{i*} \;=\; \beta_i(1 + r_{\tau+1}^*)C_\tau^{i*} - (1-\beta_i)(1+r_{\tau+1}^*)\frac{n}{1+n}S_\tau^{i*}\,, \qquad \tau \geq 0\,. \tag{5}$$

Recall that in the Ramsey–Becker model with population growth, newly entered agents are members of a growing dynasty, and the average consumption of type i agents with positive savings grows at the rate $\beta_i(1 + r_{\tau+1}^*)$. However, in the Becker–Weil model, newly entered agents start with zero endowments. Therefore, the average consumption is lower by the amount that newly entered agents would have consumed out of the transfers from the current population.

The value of these missing transfers (shown in Fig. 1 by arrows present in panel (a) but absent in panel (b)) is exactly $nS_\tau^{i*}/(1+n)$, and the marginal propensity to consume under log-utility is $1-\beta_i$.

A stationary equilibrium in the Becker–Weil model is defined as follows.

Definition 2. *A tuple* $\{(C^{i*}, S^{i*})_{i=H,L}, k^*, r^*, w^*\}$ *is a* Becker–Weil stationary equilibrium *if the sequence* $\{(C_\tau^{i*}, S_\tau^{i*})_{i=H,L}, k_{\tau+1}^*, r_\tau^*, w_\tau^*\}_{\tau=0}^\infty$, *where for each* $i = H, L$ *and for each* $\tau \geq 0$, $C_\tau^{i*} = C^{i*}$, $S_\tau^{i*} = S^{i*}$, $r_\tau^* = r^*$, $w_\tau^* = w^*$, *and* $k_{\tau+1}^* = k^*$, *is an equilibrium in the Becker–Weil model starting from* $(S^{H*}/(1+n), S^{L*}/(1+n))$.

Stationarity in Definition 2 refers only to *average* values. In a stationary equilibrium, in each period, after the entry of new agents, the average consumption and capital stock remain the same for each type. Individual consumption and savings levels, $\{c_t^{\tau,i*}, s_t^{\tau,i*}\}_{t=\tau}^\infty$, which solve problem (1) at given $\{r^*, r^*, \ldots\}$ and $\{w^*, w^*, \ldots\}$ starting from $s_{-1}^{0,i} = S^{i*}/(1+n)$ and $s_{\tau-1}^{\tau,i} = 0$ for $\tau \geq 1$, need not be constant in a stationary equilibrium.

In order to characterize a Becker–Weil stationary equilibrium, we consider utility functions with constant elasticity of intertemporal substitution together with a production function with constant elasticity of substitution greater than 1. Specifically, we make the following assumptions about preferences and technology.[4]

Assumption 1. *All agents have the same* isoelastic *instantaneous utility function:*

$$u_i(c) \;=\; u(c) \;=\; \frac{c^{1-\frac{1}{\eta}} - 1}{1-\frac{1}{\eta}}\,,$$

where $\eta > 0$ *is the constant elasticity of intertemporal substitution.*

Assumption 2. *The production function in intensive form* $f(k)$ *is* CES*:*

$$f(k) \;=\; A\left(ak^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{\sigma}{\sigma-1}}\,,$$

where $0 < a < 1$ *and* $A > 0$. *Moreover, capital and labor are substitutes:* $\sigma \geq 1$.

In what follows, we impose Assumptions 1 and 2. Denote the *capital income share* by $\alpha(k) \equiv kf'(k)/f(k)$. Also, for $i = H, L$, let k_{β_i} be the capital stock corresponding to the modified golden rule for the discount factor β_i, that is, $f'(k_{\beta_i}) = 1/\beta_i$.

[4] We relax these assumptions in Section 6, where we prove the existence of a Becker–Weil stationary equilibrium for any neoclassical production function and for isoelastic utility functions with different elasticities of intertemporal substitution for different agent types.

We show that there exists a unique stationary equilibrium. A stationary equilibrium is one of two types: a *Ramsey conjecture stationary equilibrium* in which only the most patient agents own the entire capital stock, or a *non-degenerate stationary equilibrium* in which both types of agents own positive amounts of capital. The following theorem provides the exact threshold between the two types of equilibrium in terms of the parameters of the model.

Theorem 1. *There exists a unique Becker–Weil stationary equilibrium. The stationary per capita capital stock k^* increases with β_H, non-decreases with β_L, and decreases with n.*

1. *Suppose $1+n > (\beta_H/\beta_L)^\eta$, and the parameters of the model are such that*

$$\frac{(\beta_H)^\eta - (\beta_L)^\eta}{(1+n)(\beta_L)^\eta - (\beta_H)^\eta} \, \frac{1/\alpha(k_{\beta_L}) - 1}{1-\beta_L} \;<\; \frac{1}{\pi_H}\,, \tag{6}$$

   where $\alpha(k_{\beta_L}) = k_{\beta_L} f'(k_{\beta_L})/f(k_{\beta_L}) = a^\sigma (A\beta_L)^{\sigma-1}$. In the stationary equilibrium both types of agents own positive amounts of capital: $S^{H*} > S^{L*} > 0$, and $k^* < k_{\beta_L}$.

2. *Suppose $1+n \le (\beta_H/\beta_L)^\eta$, or $1+n > (\beta_H/\beta_L)^\eta$ and condition (6) does not hold. In the stationary equilibrium the most patient agents own the entire capital stock and the least patient agents do not save: $S^{H*} = (1+n)k^*/\pi_H$, $S^{L*} = 0$, and $k_{\beta_L} \le k^* < k_{\beta_H}$.*

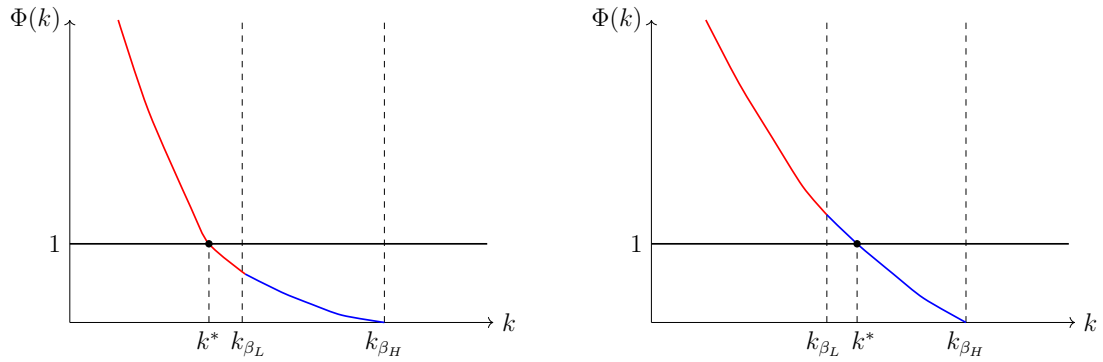
Proof. See Appendix A. ■

Theorem 1 implies that under condition (6), a unique Becker–Weil stationary equilibrium is non-degenerate, and both the most patient agents and the least patient agents own positive amounts of capital. Therefore, in the Ramsey model with heterogeneous agents and the OLG demographic structure, *the Ramsey conjecture may not hold*. The entry of new agents without endowments prevents the most patient agents from holding the entire capital stock. Intuitively, since newly entering agents are not linked to the existing ones, the current population saves less (compared to the Ramsey demographic structure), keeping the interest rate high and allowing the least patient agents to accumulate capital.

Theorem 1 deserves several comments. First, the role of condition (6) is illustrated in Fig. 2. The stationary capital stock k^* is a solution to

$$\Phi(k) \;\equiv\; \pi_H \frac{S^{H*}(k)}{(1+n)k} + \pi_L \frac{S^{L*}(k)}{(1+n)k} \;=\; 1\,,$$

where $S^{i*}(k)$ are average savings of type i agents as a function of capital stock (via interest rate and wage rate). Note that $S^{H*}(k_{\beta_H}) = 0$, and $S^{L*}(k) \equiv 0$ for all $k \geq k_{\beta_L}$. We show that when $\sigma \geq 1$, the saving-capital ratio $\Phi(k)$ strictly decreases with k, implying there is a unique k^* such that $\Phi(k^*) = 1$. Condition (6) requires that $\Phi(k_{\beta_L}) < 1$, in which case $k^* < k_{\beta_L}$, so both types of agents own capital in the stationary equilibrium (panel (a) on the left). If condition (6) does not hold, then $\Phi(k_{\beta_L}) \geq 1$, in which case $k^* \geq k_{\beta_L}$, implying that only the most patient agents own capital in the stationary equilibrium (panel (b) on the right).



(a) Non-degenerate equilibrium

(b) Ramsey conjecture equilibrium

Figure 2: Non-degenerate (a) and Ramsey conjecture (b) stationary equilibria.

Whether condition (6) holds depends on the parameters of the model as well as on the properties of the utility and production functions. In particular, condition (6) is satisfied in the following cases. First, similar discount factors: β_H is close to β_L. When heterogeneity in patience is low, the model with heterogeneous agents is close to the model with a single discount factor analyzed by Weil (1989). Second, high population growth rate: n is sufficiently large. The entry of more agents with zero endowments each period erodes per capita capital, raises the interest rate, and strengthens the incentives for the least patient agents to save. Third, low elasticity of intertemporal substitution: η is small enough. The less willing agents are to substitute consumption intertemporally, the smaller the effect of the discount factor on saving decisions. Finally, high capital income share (evaluated at k_{β_L}): $\alpha(k_{\beta_L}) = a^{\sigma}(A\beta_L)^{\sigma-1}$ is close to 1. The higher the capital income share, the greater the marginal benefit of saving for the least patient agents.

Second, there are multiple sources of *Pareto inefficiency* of equilibria in the Becker–Weil model. Since capital markets are incomplete under borrowing constraints, any equilibrium in the Ramsey–Becker setting is not Pareto-optimal. This feature is shared by both the Becker–Weil model and the Ramsey–Becker

model with population growth.[5] Furthermore, equilibrium in the Becker–Weil model can be *dynamically inefficient* and characterized by overaccumulation of capital. This feature is shared with typical OLG models but stands in stark contrast to the Ramsey–Becker model with population growth.[6]

Recall that equilibrium is dynamically inefficient if the stationary capital stock k^* exceeds the golden rule capital stock k_{GR}, which satisfies $f'(k_{GR}) = 1 + n$, or, equivalently, if the stationary interest rate satisfies $r^* < n$. In the Becker–Weil model, even though r^* increases with n, for a sufficiently high population growth rate, an equilibrium is inefficient. Intuitively, as in any OLG setting, dynamic inefficiency arises from a pecuniary externality caused by the actions of previous generations on the current generation: the more previous generations save, the lower the interest rate becomes, which incentivizes the current generation to save even more, eventually leading to overaccumulation of capital. Since there are infinitely many goods and generations, pecuniary externalities are not negligible.

We can provide simple sufficient conditions for dynamic efficiency. A non-degenerate equilibrium (where $k^* < k_{\beta_L}$) is efficient when $(1+n)\beta_L < 1$. Similarly, a Ramsey conjecture equilibrium (where $k^* < k_{\beta_H}$) is efficient when $(1+n)\beta_H < 1$. In particular, efficiency always holds when $n = 0$, which corresponds to the case where the Becker–Weil model coincides with the Ramsey–Becker model. However, these conditions need not hold in general. For instance, in the innocuous case where $(1 + n)\beta_L > 1$, a Ramsey conjecture equilibrium is dynamically inefficient, and the most patient agents overaccumulate capital.

Finally, in any stationary equilibrium, the stationary capital stock k^* increases with the discount factor of the agent types who save in that equilibrium. Specifically, k^* increases with β_H, reflecting the usual intuition that more patient societies accumulate more capital. Also, in both types of stationary equilibrium, k^* decreases with the population growth rate n. A higher number of new agents entering the economy without endowments implies more "missing" intergenerational transfers and a lower capital stock.

[5] In Section 6, we study a model with complete capital markets (without borrowing constraints) in the Bewley (1982) setting, which eliminates this source of inefficiency.

[6] The absence of capital overaccumulation is typical of Ramsey-type models with infinitely-lived agents. In the Ramsey–Becker setting, Becker and Mitra (2012) show that equilibrium paths of aggregate capital and consumption are dynamically efficient, despite being non-optimal.

4 Fiscal policies and capital accumulation

Theorem 1 suggests that a given economy, characterized by certain discount factors and population growth rate, admits either a Ramsey conjecture stationary equilibrium or a non-degenerate stationary equilibrium, depending on condition (6). In this section, we show how standard fiscal policies can shift the economy from a Ramsey-conjecture-type equilibrium to a non-degenerate equilibrium, and study the implications for the stationary capital stock.

4.1 Public debt

Consider the Becker–Weil model with public debt. The main assumptions remain the same as in the baseline Becker–Weil model, but now we introduce a public sector represented by the per capita level of public debt b. Debt is issued in the form of government bonds and financed through a per capita lump-sum tax θ imposed on all agents in all periods. Assuming no government spending, the government's budget constraint in each period τ is

$$(1+n)b_{\tau+1} + \theta_\tau \;=\; (1+r_\tau)b_\tau\,. \tag{7}$$

We assume that the government controls the debt-to-output ratio, $\chi = b/f(k)$. For any given $\chi \geq 0$, the government chooses the corresponding tax rate from (7).

Each agent of type $i = H, L$ entered in period τ solves the following problem:

$$\begin{aligned} \max_{\{c_t, s_t\}_{t=\tau}^{\infty}} \sum_{t=\tau}^{\infty} (\beta_i)^{t-\tau} u_i(c_t)\,, \qquad \text{s. t.} \qquad & c_t + s_t \;=\; (1+r_t)s_{t-1} + w_t - \theta_t, \\ s_t \;\geq\; 0, \quad t \geq \tau\,, \qquad & s_{\tau-1} \;=\; s_{\tau-1}^{\tau,i}\,, \end{aligned} \tag{8}$$

where the sequences of interest rates $\{r_t\}_{t=\tau}^{\infty}$, wage rates $\{w_t\}_{t=\tau}^{\infty}$ and lump-sum taxes $\{\theta_t\}_{t=\tau}^{\infty}$ are taken as given.

The definition of a general equilibrium in the Becker–Weil model with public debt is a modification of Definition 1, taking into account that capital stock and public debt are perfect substitutes for agents as wealth owners.

Definition 3. *A sequence* $\{(C_\tau^{i*}, S_\tau^{i*})_{i=H,L}, k_{\tau+1}^*, b_{\tau+1}^*, r_\tau^*, w_\tau^*\}_{\tau=0}^{\infty}$ *is an* equilibrium in the Becker–Weil model with public debt *starting from* $(S_{-1}^{H*}, S_{-1}^{L*})$ *with* $k_0^* + b_0^* = \pi_H S_{-1}^{H*} + \pi_L S_{-1}^{L*}$*, if in each period* $\tau \geq 0$*:*

1. *Each agent of type* $i = H, L$ *entered in period* τ *determines individual consumption and savings* $\{c_t^{\tau,i*}, s_t^{\tau,i*}\}_{t=\tau}^{\infty}$ *by solving problem* (8) *at given*

$\{r_t^*\}_{t=\tau}^{\infty}$ *and* $\{w_t^*\}_{t=\tau}^{\infty}$ *starting from initial savings* $s_{-1}^{0,i} = S_{-1}^{i*}$ *and* $s_{\tau-1}^{\tau,i} = 0$ *for* $\tau \geq 1$*, given the sequence of taxes* $\{\theta_t^*\}_{t=\tau}^{\infty}$ *determined by* (7)*:*

$$\theta_\tau^* \;=\; \chi\left[(1+r(k_\tau^*))f(k_\tau^*) - (1+n)f(k_{\tau+1}^*)\right]\,;$$

2–3. *Conditions* 2–3 *from Definition 1 hold;*

4. *Aggregate savings are equal to the sum of per capita capital stock and debt:*

$$(1+n)\left(k_{\tau+1}^* + b_{\tau+1}^*\right) \;=\; \pi_H S_\tau^{H*} + \pi_L S_\tau^{L*}\,;$$

5. *The per capita debt is given by* $b_\tau^* \;=\; \chi f(k_\tau^*)$*.*

A Becker–Weil stationary equilibrium with public debt is naturally defined following Definition 2 from the baseline model. Note that when the equilibrium is dynamically inefficient ($r^* < n$), it follows from (7) that the government can finance debt without imposing taxes. Therefore, we focus only on dynamically efficient equilibrium, in particular, assuming that $(1+n)\beta_H < 1$.

The following theorem shows that a sufficiently high debt-to-output ratio in the Becker–Weil model leads to a non-degenerate stationary equilibrium.

Theorem 2. *There is a maximum debt-to-output ratio* $\overline{\chi}$*, such that for any* $\chi < \overline{\chi}$*, there exists a stationary equilibrium. The stationary capital stock* k^* *for given* χ *is lower than the stationary capital stock without debt (*$\chi = 0$*).*

Suppose $1/\beta_H > 1 + n > (\beta_H/\beta_L)^\eta$ *and condition* (6) *does not hold. There is a threshold debt-to-output ratio* χ^**, such that for any given* $\chi \in (\chi^*, \overline{\chi})$*, any stationary equilibrium is non-degenerate.*

Proof. See Appendix B. ■

Theorem 2 yields several key insights. First, in any stationary equilibrium, public debt crowds out private capital and reduces the stationary capital stock, implying that *Ricardian equivalence fails* in the Becker–Weil model.[7] Intuitively, since newly entering agents are not linked to existing agents, the current population does not account for taxes that will be imposed on future generations and fails to save enough to completely offset the effects of public debt.

This result is consistent with the Weil (1989) model, in which all agents share the same discount factor, but stands in stark contrast to existing models with

[7] The uniqueness of a stationary equilibrium with public debt is not guaranteed. In particular, the possibility for both a non-degenerate stationary equilibrium and a Ramsey conjecture stationary equilibrium to coexist for a given debt-to-output ratio cannot be ruled out.

heterogeneous agents, where both Ricardian equivalence and the Ramsey conjecture hold in the long run (e.g., Mankiw, 2000; Michel and Pestieau, 2005). In particular, in the Ramsey–Becker model with population growth, the introduction of public debt lowers stationary consumption but leaves the stationary capital stock unchanged, precisely because the most patient dynasty internalizes the welfare of all its descendants. Thus, our results highlight the relationship between Ricardian equivalence (debt neutrality), the Ramsey conjecture (degenerate wealth distribution), and the OLG demographic structure (entry of new generations not included in the economic calculus of existing ones).

Second, when the baseline Becker–Weil model admits a unique Ramsey conjecture stationary equilibrium, any stationary equilibrium with a sufficiently high debt-to-output ratio is non-degenerate. In terms of comparative statics, introducing public debt financed by a lump-sum tax enables the economy to shift from a Ramsey conjecture stationary equilibrium to a non-degenerate one. Intuitively, the presence of public debt as a perfect substitute to capital stock decreases agents' holdings of capital, while a lump-sum tax reduces their lifetime incomes. In terms of Fig. 2, the saving-capital ratio $\Phi(k)$ shifts downward for all values of k. As a result, the most patient agents save less, raising the stationary interest rate and allowing the least patient agents to save as well.

4.2 Income taxation and public spending

A lump-sum tax plays a key role in the shifting result in the model with public debt. To highlight this effect, we consider the Becker–Weil model with different taxes on capital and labor income. We show that labor income tax enables the economy to shift from a Ramsey conjecture stationary equilibrium to a non-degenerate stationary equilibrium, while capital tax or wealth tax preserve a Ramsey-conjecture-type equilibrium.

Consider the Becker–Weil model with income taxation. Each agent of type $i = H, L$ entered in period τ solves the following problem:

$$\max_{\{c_t, s_t\}_{t=\tau}^{\infty}} \sum_{t=\tau}^{\infty} (\beta_i)^{t-\tau} u_i(c_t)\,, \quad \text{s. t.} \quad c_t + s_t \;=\; (1-\theta_K)(1+r_t)s_{t-1} + (1-\theta_N)w_t\,,$$

$$s_t \;\geq\; 0, \quad t \geq \tau\,, \qquad\qquad s_{\tau-1} \;=\; s_{\tau-1}^{\tau,i}\,,$$

where $\theta_K \geq 0$ is the capital tax rate and $\theta_N \geq 0$ is the labor income tax rate applied to all agents in all periods.

Capital and labor income taxes are used to finance an endogenous level of per

capita public spending g. Given θ_K and θ_N, government spending in each period τ is determined by a balanced budget rule:

$$g_\tau \;=\; \theta_K(1+r_\tau)k_\tau + \theta_N w_\tau\,.$$

We assume that the level of public spending g_τ affects neither agents' utilities nor production function (cf. Bosi and Seegmuller, 2010).

The remaining assumptions are the same as in the baseline model. The following theorem shows that, in the Becker–Weil model with income taxation, a shift from a Ramsey conjecture stationary equilibrium to a non-degenerate stationary equilibrium is possible only under a sufficiently high labor income tax.

Theorem 3. *For any given pair $(\theta_K, \theta_N) \in [0,1) \times [0,1)$ there exists a unique stationary equilibrium. The stationary capital stock k^* decreases with θ_N, and for $\eta \geq 1$ decreases with θ_K.*

Suppose $1 + n > (\beta_H/\beta_L)^\eta$ and condition (6) does not hold. For any given θ_K, there is a threshold $\theta_N^*(\theta_K)$ such that a unique stationary equilibrium is non-degenerate if and only if $\theta_N > \theta_N^*(\theta_K)$. Moreover, $\theta_N^*(\theta_K)$ is concave, with $\theta_N^*(0) \geq 0$ and $\theta_N^*(1) = 1$.

Proof. See Appendix C. ■

Theorem 3 states that if the baseline Becker–Weil model without taxes admits a unique Ramsey conjecture stationary equilibrium, there exists a region in the two-dimensional tax space (θ_K, θ_N) where any combination of capital and labor income taxes ensures a unique non-degenerate stationary equilibrium. This admissible region of tax rates is shown in Fig. 3. It follows that for shifting from a Ramsey-conjecture-type to a non-degenerate stationary equilibrium, the labor income tax rate should always exceed the capital tax rate.

To illustrate Theorem 3, suppose the baseline Becker–Weil model without taxes admits a unique Ramsey conjecture stationary equilibrium, and consider three important special cases. First, consider only a *labor income tax* θ_N (with $\theta_K = 0$, corresponding to the vertical axis in Fig. 3). In this case, for a sufficiently high $\theta_N > \theta_N^*(0) \geq 0$, the Becker–Weil model with a labor income tax admits a unique non-degenerate equilibrium. Thus, introducing a labor income tax enables the economy to shift from a Ramsey conjecture stationary equilibrium to a non-degenerate stationary equilibrium.

Since labor supply is inelastic, the labor income tax is equivalent to a lump-sum tax, following the same intuition as in the model with public debt. The current population ignores taxes levied on future generations and does not fully

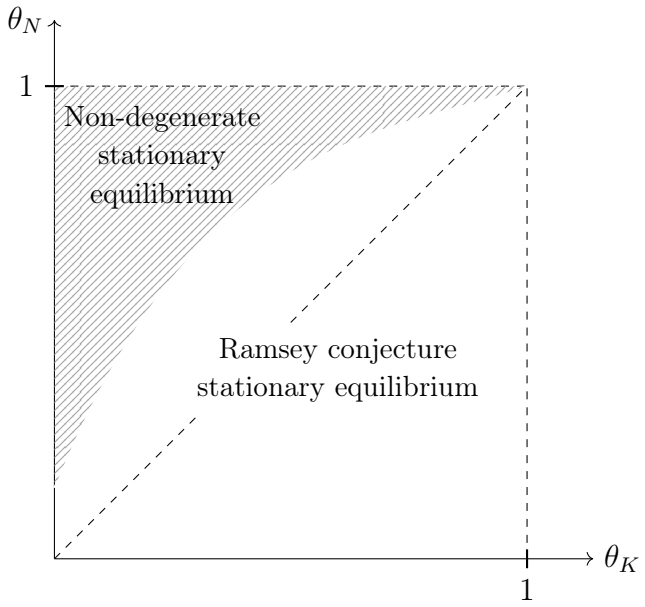


Figure 3: Effect of capital tax and labor income tax on the stationary equilibrium.

internalize the cost of financing unproductive public spending. The labor income tax reduces agents' lifetime incomes (for any given interest rate), causing the most patient agents to save less. This lowers the stationary capital stock and raises the equilibrium interest rate, allowing the least patient agents to save as well.

Second, consider only a *capital tax* θ_K (with $\theta_N = 0$, corresponding to the horizontal axis in Fig. 3). As usual, the response to capital taxation is determined by the elasticity of intertemporal substitution (see, e.g., Caballé, 1998; Saez, 2013). When $\eta > 1$, the substitution effect dominates the income effect, so a lower rate of return reduces individual savings and the stationary capital stock falls. The same holds for log-utility ($\eta = 1$), while when $\eta < 1$ the opposite outcome is possible.

At the same time, for any η, a capital tax does not change the type of stationary equilibrium. This is because it reduces the incentives to save for both types of agents and raises the threshold interest rate at which the least patient agents begin to accumulate capital. If the economy is in a Ramsey conjecture stationary equilibrium, the introduction of a capital tax keeps the stationary interest rate below this new threshold, so that only the most patient agents can save.

Third, consider a wealth tax, $\theta_K = \theta_N$ (corresponding to the 45-degree line in Fig. 3). The effect of a wealth tax is similar to that of a capital tax: it does not change the type of stationary equilibrium. Intuitively, a labor income tax reduces lifetime incomes, and a capital tax reduces incentives to save by shifting the thresholds, so that the stationary interest rate is never high enough for the least patient agents to save.

4.3 Redistributive taxation

In the Becker–Weil model with income taxation and unproductive public spending, tax revenues are effectively removed from the economy. We show that our results about capital taxation remain valid also when tax revenues are transferred back to the agents. We consider the Becker–Weil model with redistributive capital taxation and show that the economy shifts to a non-degenerate stationary equilibrium only if capital is subsidized.

Consider the Becker–Weil model with redistributive taxation. Each agent of type $i = H, L$ entered in period τ solves the problem

$$\max_{\{c_t, s_t\}_{t=\tau}^{\infty}} \sum_{t=\tau}^{\infty} (\beta_i)^{t-\tau} u_i(c_t)\,, \quad \text{s. t.} \quad c_t + s_t \;=\; (1-\vartheta)(1+r_t)s_{t-1} + w_t + \vartheta_t^{\tau,i}\,,$$

$$s_t \;\geq\; 0, \quad t \geq \tau\,, \qquad\qquad s_{\tau-1} \;=\; s_{\tau-1}^{\tau,i}\,.$$

In each period, the government imposes a capital tax ϑ on each agent and redistributes the revenue back to the agent as a lump-sum subsidy $\vartheta_t^{\tau,i}$, which in equilibrium is equal to $\vartheta_t^{\tau,i} = \vartheta(1+r_t)s_{t-1}^{\tau,i}$. If ϑ is negative, it is interpreted as a capital subsidy, financed through an individual lump-sum tax of the same form. Thus, individual income does not change after redistribution.

The remaining assumptions are identical to those in the baseline Becker–Weil model. The following theorem shows that, in the Becker–Weil model with redistributive taxation, a shift from a Ramsey-conjecture-type stationary equilibrium to a non-degenerate stationary equilibrium occurs only under a sufficiently large capital subsidy.

Theorem 4. *There is a maximum capital tax $\overline{\vartheta} > 0$ and a maximum capital subsidy $\underline{\vartheta} < 0$, such that for any $\vartheta \in (\underline{\vartheta}, \overline{\vartheta})$ there exists a unique stationary equilibrium. For $\eta \geq 1$, the stationary capital stock k^* decreases with ϑ.*

Suppose $1+n > (\beta_H/\beta_L)^{\eta}$ and condition (6) does not hold. There is a threshold capital subsidy $\vartheta^* < 0$ such that a unique stationary equilibrium is non-degenerate if and only if $\vartheta \in (\underline{\vartheta}, \vartheta^*)$.

Proof. See Appendix D. ■

Theorem 4 implies that if the baseline Becker–Weil model without taxes admits a unique Ramsey conjecture stationary equilibrium, then a shift to a non-degenerate stationary equilibrium is possible only in the presence of a capital *subsidy* (financed by a corresponding lump-sum tax). As in Theorem 3, a capital tax cannot change the type of stationary equilibrium.

Intuitively, under redistributive capital taxation two effects are at play. First, the *threshold-shifting* effect: a higher capital tax reduces the incentives to save for both types of agents, raising the threshold interest rate at which the least patient agents are willing to save and moving the saving-capital ratio $\Phi(k)$ in Fig. 2 downward. Second, the *general equilibrium average savings* effect: although redistribution leaves individual income unchanged, the redistributed tax revenue alters average savings, moving $\Phi(k)$ upward. The net effect on the stationary capital stock k^* depends on the elasticity of intertemporal substitution η, but the threshold shift preserves the Ramsey-conjecture-type equilibrium. By the same argument, our results remain qualitatively similar under uniform redistribution if each agent receives an equal per capita transfer $\vartheta(1 + r_t)k_t$.

By contrast, under a capital subsidy, both effects reverse. A higher subsidy lowers the threshold return on capital required for the least patient agents to start saving, while the lump-sum tax used to finance the subsidy reduces the saving-capital ratio. Although the effect on k^* depends on η, the threshold shift implies that a sufficiently large subsidy can move the economy from a Ramsey conjecture stationary equilibrium to a non-degenerate stationary equilibrium.

5 Intra- and intergenerational inequality

Theorems 2–4 show that public debt, a labor income tax, or a capital subsidy (for $\eta \geq 1$) change the type of stationary equilibrium, allowing less patient agents to save. In this section, we study the effect of fiscal policies on the wealth distribution and analyze the trade-off between income and inequality in a stationary equilibrium.

In the Becker–Weil model, agents are heterogeneous in two dimensions. They differ both in age (in each period τ, agents who entered in periods 0 through τ coexist in the economy) and in type (within each generation, agents have different discount factors). As a result, there is *between-type (intragenerational)* inequality among different agent types from the same generation, and *within-type (intergenerational)* inequality among different generations of agents of the same type. For clarity, we study these two dimensions of inequality separately.

5.1 Between-type inequality

A natural measure of *intragenerational* inequality is the Gini index of the average savings S^{i*} of different agent types. Since all agents receive the same wage, wealth inequality is primarily driven by capital inequality. Furthermore, the average

consumption of type i agents is given by $C^{i*} = (r^* - n)S^{i*}/(1+n) + w^*$. When the equilibrium is dynamically efficient ($r^* > n$), $S^{H*} > S^{L*}$ implies $C^{H*} > C^{L*}$, resulting in a similar pattern of consumption inequality.

For two groups of agents with population shares π_H and π_L, characterized by average savings S^{H*} and S^{L*} (with $S^{H*} > S^{L*}$), the Gini index of capital distribution in a stationary equilibrium is constant over time and given by

$$G^* = \frac{\pi_L \pi_H \left(S^{H*} - S^{L*}\right)}{\pi_H S^{H*} + \pi_L S^{L*}}. \tag{9}$$

The following proposition characterizes the dependence of the stationary (between-type) Gini index on the stationary capital stock and the population growth rate.

Proposition 1. *The Gini index in a stationary equilibrium increases with k^* for $k^* < k_{\beta_L}$, and is constant for $k_{\beta_L} \leq k^* < k_{\beta_H}$. Moreover, there is a threshold value of population growth $\overline{n}$, such that the Gini index is constant for $n \leq \overline{n}$, and decreases with n for $n > \overline{n}$.*

Proof. See Appendix E. ■

Proposition 1 implies that in a Ramsey conjecture stationary equilibrium, the Gini index is independent of the stationary capital stock. Since only type H agents own capital, we have $S^{L*} = 0$ and $G^* = 1 - \pi_H$. For sufficiently high values of the stationary capital stock ($k_{\beta_L} \leq k^* < k_{\beta_H}$), further increases in k^* make saving even less attractive for the least patient agents and does not alter the degenerate distribution of wealth. In a non-degenerate stationary equilibrium, we have $S^{H*} > S^{L*} > 0$, and the Gini index increases with k^*. Thus, for relatively low values of the stationary capital stock ($k^* \leq k_{\beta_L}$), there is a trade-off between income and inequality: an increase in k^* raises inequality. Intuitively, since the most patient agents have a higher propensity to save, a higher k^* corresponds to a larger ratio S^{H*}/S^{L*}, thereby increasing inequality.

Furthermore, we study the dependence of between-type inequality on the population growth rate n. Clearly, the Gini index in a Ramsey conjecture stationary equilibrium is independent of n. At the same time, the Gini index in a non-degenerate stationary equilibrium decreases with n. Indeed, G^* increases with k^*, and, by Theorem 1, k^* decreases with n. Loosely speaking, the Becker–Weil model establishes a negative correlation between the rate of demographic expansion and between-type inequality: a higher n leads to a more equal distribution of capital between the most patient and least patient agents in a non-degenerate stationary equilibrium.

Proposition 1 allows us to analyze the impact of fiscal policies on between-type inequality. Suppose the economy is in a Ramsey conjecture stationary equilibrium. According to Theorem 2, when the debt-to-output ratio is sufficiently high, the economy shifts to a non-degenerate stationary equilibrium. As in clear from Proposition 1, this shift reduces the Gini index. Thus, introducing public debt creates a *trade-off* between intragenerational inequality and income: a higher debt-to-output ratio lowers both between-type inequality and the stationary capital stock.

It follows from Theorem 3 that a labor income tax implies a similar trade-off. By allowing the economy to shift to a non-degenerate stationary equilibrium, a labor income tax lowers the Gini index and reduces inequality, but this comes at the cost of a lower stationary capital stock. By contrast, neither a capital tax nor a wealth tax affects between-type inequality, since these policies do not enable the economy to leave the Ramsey-conjecture-type equilibrium.

Finally, Theorem 4 suggests that a particularly effective policy tool for addressing between-type inequality in the Becker–Weil model is a capital subsidy financed by a lump-sum tax. Its introduction both lowers the Gini index (by shifting the economy to a non-degenerate stationary equilibrium) and raises the stationary capital stock (at least when $\eta \geq 1$ so that agents are sufficiently willing to substitute consumption intertemporally).

5.2 Within-type inequality

Consider now *intergenerational* inequality. In a Ramsey conjecture stationary equilibrium, the interest rate prevents the least patient agents from saving: $1/\beta_H < 1+r^* < 1/\beta_L$. At the individual level, all type L agents from different generations are perfectly equal: they do not save and in each period consume their wages, $c_t^{\tau,L*} = w^*$ and $s_t^{\tau,L*} = 0$ for all $t \geq \tau$ and all $\tau \geq 0$.

At the same time, individual savings of type H agents grow at a positive rate without bound in a stationary equilibrium. For a type $i = H$ agents entered in period τ, the dynamics of savings are given by

$$s_t^{\tau,i*} \;=\; (1+\lambda_i^*)s_{t-1}^{\tau,i*} + \frac{w^*}{r^*}\lambda_i^*\,, \qquad t \geq \tau\,, \tag{10}$$

where $1+\lambda_i^* = [\beta_i(1+r^*)]^\eta$ is the growth rate of individual consumption. Hence, in each period τ, there is inequality among type H agents from different generations: $s_\tau^{0,H*} > s_\tau^{1,H*} > \ldots > s_\tau^{\tau,H*}$. As earlier generations began accumulating capital earlier, they have higher wealth in each period, compared to later generations.

In a non-degenerate stationary equilibrium, the interest rate is sufficiently high for the least patient agents to save: $1 + r^* > 1/\beta_L$. At the individual level, each agent of type $i = H, L$, in each period makes positive savings, and the dynamics of savings for both types of agents follow (10). Earlier generations of both types of agents are relatively richer, and there is within-type intergenerational inequality for both types. For each $i = H, L$, we have $s_\tau^{0,i*} > s_\tau^{1,i*} > \ldots > s_\tau^{\tau,i*}$ for all $\tau \geq 1$.

It is useful to measure intergenerational inequality by comparing the wealth of agents of the same type from different generations at specific percentiles. A common measure of dispersion is the $P90/P10$ ratio, defined as the ratio of wealth at the 90th percentile to that at the 10th percentile. Note that the $P90/P10$ ratio in a stationary equilibrium is not constant but increases over time. The following proposition characterizes the dependence of the $P90/P10$ ratio on the stationary gross interest rate for any given period.

Proposition 2. *For any given $\tau > 1$, for the most patient agents, the $P90/P10$ ratio in a stationary equilibrium increases with $1+r^*$. For the least patient agents, the $P90/P10$ ratio in a stationary equilibrium is constant for $1/\beta_H < 1+r^* < 1/\beta_L$ and increases with $1 + r^*$ for $1 + r^* > 1/\beta_L$.*

Proof. See Appendix F. ■

Proposition 2 relies on the fact that if type i agents own capital in the stationary equilibrium, then in any period τ, the $P90/P10$ ratio for type i increases with λ_i^*, which in turn rises with $1 + r^*$. Intuitively, the faster agents accumulate capital, the larger the gap between richer generations who began accumulating earlier and poorer generations who enter without endowments.

As an illustration, suppose that in some period τ, the wealth at $P90$ corresponds to generation 1, while the wealth at $P10$ corresponds to generation τ. Since both generations 1 and τ start with zero initial savings, it is easily seen from (10) that

$$\frac{s_\tau^{1,i*}}{s_\tau^{\tau,i*}} = \frac{(1+\lambda_i^*)^\tau - 1}{\lambda_i^*},$$

which clearly increases with the growth rate λ_i^*.[8]

We apply Proposition 2 to analyze the impact of fiscal policies on within-type inequality. Suppose the economy is in a Ramsey conjecture stationary equilibrium. It follows from Theorems 2 and 3 that a sufficiently high debt-to-output ratio

[8] The actual 90th and 10th percentiles depend on the population growth rate n and period τ. We show that for any two generations of the same type i, the savings ratio between earlier and later generations increases with λ_i^*. Hence, our results hold for any intergenerational inequality measure expressed as a percentile ratio (e.g., the $P90/P10$ ratio, the $P80/P20$ ratio, etc.).

or labor income tax rate not only reduces the stationary capital stock k^* but also *increases within-type inequality* for both types of agents. Both policies shift the economy to a non-degenerate stationary equilibrium with a higher stationary interest rate r^*, which raises λ_i^* and thereby the $P90/P10$ ratio. Intuitively, with a higher interest rate, both types of agents accumulate capital faster in a non-degenerate stationary equilibrium. Hence, in any given period, generations that began accumulating capital earlier are relatively richer than newly entered ones, compared to the Ramsey conjecture equilibrium.

By contrast, Theorems 3 and 4 imply that a capital tax (at least for $\eta \geq 1$), regardless of whether revenues are used to finance unproductive government spending or redistributed back to agents, *reduces within-type inequality* for the most patient agents. The reason is that when $\eta \geq 1$, the after-tax gross interest rate, $(1-\vartheta)(1+r^*)$, decreases with the tax rate ϑ. Since the growth rate of individual consumption, λ_i^*, depends on this after-tax return, λ_i^* falls even though the marginal product $1+r^*$ increases. At the same time, within-type inequality for the least patient agents is unaffected, because the capital tax preserves a Ramsey conjecture stationary equilibrium. Thus, capital taxation creates a *trade-off* between intergenerational inequality and income: it reduces within-type inequality at the cost of a lower stationary capital stock.

By the same argument, a capital subsidy financed by a lump-sum tax implies the opposite trade-off: it increases both the stationary capital stock and within-type inequality for both types of agents. The subsidy strengthens incentives to save, leading to a higher stationary capital stock in a non-degenerate equilibrium (for $\eta \geq 1$). However, the after-subsidy gross interest rate, $(1+\vartheta)(1+r^*)$, is higher than without the subsidy, which accelerates capital accumulation for both types of agents and raises within-type inequality.

Table 1 summarizes the impact of fiscal policies on the stationary capital stock k^*, the Gini index G^* and the $P90/P10$ ratio discussed in Sections 4 and 5.

Policy \ Variable	k^*	G^*	$P90/P10$
Public debt	$\downarrow$	$\downarrow$	$\uparrow$
Labor income tax	$\downarrow$	$\downarrow$	$\uparrow$
Capital subsidy ($\eta \geq 1$)	$\uparrow$	$\downarrow$	$\uparrow$
Capital tax ($\eta \geq 1$)	$\downarrow$	$=$	$\downarrow$

Table 1: Fiscal policies in a stationary equilibrium.

6 Extensions and modifications

In this section, we consider two modifications of the Becker–Weil model: a general setting with any neoclassical production function and many types of heterogeneous agents who differ in their elasticities of intertemporal substitution; and a model with complete capital markets (without borrowing constraints). We show that our key findings remain valid: both extensions admit several types of stationary equilibrium, a Ramsey-conjecture-type and a non-degenerate-type.

6.1 Many types of agents

Consider a model with I types of heterogeneous agents and a general neoclassical production function. Each agent of type $i = 1, \ldots, I$ is characterized by a constant discount factor β_i, and agents are ordered in terms of their patience: $1 > \beta_1 > \ldots > \beta_I > 0$. Agents have *isoelastic* instantaneous utility functions with different elasticities $\eta_i > 0$:

$$u_i(c) \;=\; \frac{c^{1-\frac{1}{\eta_i}} - 1}{1 - \frac{1}{\eta_i}}\,.$$

New agents of each type enter in the same proportion, so the share of each type of agents, π_i, is constant over time. Technology is described by a neoclassical production function in intensive form, $f(k)$, satisfying the standard assumptions. The remaining assumptions are the same as in the baseline Becker–Weil model.

Definitions 1–2 can be naturally reformulated to include many types of agents. In this case, the equilibrium condition which determines the per capita capital stock takes the form $(1+n)k^*_{\tau+1} = \sum_{i=1}^{I} \pi_i S^{i*}_\tau$. We show that, for certain parameter values, there exists a stationary equilibrium in which any number of types of agents own positive amounts of capital. The type of stationary equilibrium is determined by agents' heterogeneity in patience and not by their heterogeneity in elasticities. If agent j owns a positive amount of capital in the stationary equilibrium, then all agents $i < j$ with higher discount factors also own capital, regardless of their elasticities of intertemporal substitution.

The following proposition provides a condition under which there exists a non-degenerate stationary equilibrium in which all agents save, and another condition under which there exists a Ramsey conjecture stationary equilibrium in which only the most patient agents own the entire capital stock.

Proposition 3. *Let* $\{(C^{i*}, S^{i*})_{i=1}^{I}, k^*, r^*, w^*\}$ *be a stationary equilibrium in the Becker–Weil model with* I *types of agents, isoelastic utilities and a general neoclassical production function.*

1. *Suppose* $1+n > \max_i\{(\beta_i/\beta_I)^{\eta_i}\}$, *and the following condition holds:*

$$\frac{1/\alpha(k_{\beta_I})-1}{1-\beta_I}\ \sum_{i=1}^{I-1}\pi_i\frac{(\beta_i)^{\eta_i}-(\beta_I)^{\eta_i}}{(1+n)(\beta_I)^{\eta_i}-(\beta_i)^{\eta_i}}\ <\ 1\,. \tag{11}$$

There exists a stationary equilibrium in which all types of agents own positive amounts of capital: $S^{1*} > \ldots > S^{I*} > 0$, *and* $k^* < k_{\beta_I}$.

2. *Suppose* $1+n \le (\beta_1/\beta_2)^{\eta_1}$, *or* $1+n > (\beta_1/\beta_2)^{\eta_1}$ *and the following condition holds:*

$$\frac{1/\alpha(k_{\beta_2})-1}{1-\beta_2}\ \frac{(\beta_1)^{\eta_1}-(\beta_2)^{\eta_1}}{(1+n)(\beta_2)^{\eta_1}-(\beta_1)^{\eta_1}}\ \ge\ \frac{1}{\pi_1}\,. \tag{12}$$

There exists a stationary equilibrium in which the most patient agents own the entire capital stock and all other agents do not save: $S^{1*} = (1+n)k^*/\pi_1$, $S^{i*} = 0$ *for all* $i \ge 2$, *and* $k_{\beta_2} \le k^* < k_{\beta_1}$.

Proof. See Appendix A. ■

In particular, Proposition 3 implies that the lower the discount factor and the higher the elasticity of intertemporal substitution of the less patient agents, the less likely condition (11) is to hold, and the more likely it is that these agents do not save in the stationary equilibrium. This observation is in line with the findings of Aguiar et al. (2025), who argue that poor hand-to-mouth consumers in the U.S. have low discount factors and high elasticities of intertemporal substitution.

Note also that for a general production function, a stationary equilibrium in the Becker–Weil model is not necessarily unique. Exploring coexistence and local stability of multiple equilibria presents a promising direction for future research.

6.2 Complete capital markets

All the previous discussions focused on the consumer problem in the Becker (1980) setting with incomplete capital markets where agents are not allowed to borrow. In this section, we describe a different version of the consumer problem with complete capital markets (without borrowing constraints) in the Bewley (1982) setting. We refer to the model with complete capital markets as the *Bewley–Weil model*.

In the Bewley–Weil model, an agent of type $i = H, L$ entered in period τ, given

initial savings $s_{\tau-1}^{\tau,i}$, solves the following problem:

$$\max_{\{c_t,s_t\}_{t=\tau}^{\infty}} \sum_{t=\tau}^{\infty}(\beta_i)^{t-\tau}u_i(c_t)\,, \qquad \text{s. t.} \quad c_t + s_t \;=\; (1+r_t)s_{t-1} + w_t, \quad t \geq \tau,$$
$$s_{\tau-1} \;=\; s_{\tau-1}^{\tau,i}\,, \qquad \lim_{t\to\infty} \frac{s_t}{(1+r_{\tau+1})\cdots(1+r_t)} \;\geq\; 0\,, \tag{13}$$

where the sequences of interest rates $\{r_t\}_{t=\tau}^{\infty}$ and wage rates $\{w_t\}_{t=\tau}^{\infty}$ are taken as given. Agents are allowed to borrow (their savings could be negative), but the No-Ponzi Game condition requires that in the limit the debt should be repaid.

The remaining assumptions are the same as in the baseline Becker–Weil model. A general equilibrium and a stationary equilibrium in the Bewley–Weil model can be defined similarly to Definitions 1 and 2. The only difference is that instead of problem (1) with borrowing constraints, each agent determines their individual consumption and savings by solving problem (13).

The following proposition characterizes a Bewley–Weil stationary equilibrium under Assumptions 1 and 2.

Proposition 4. *There exists a unique Bewley–Weil stationary equilibrium. The stationary capital stock k^* increases with β_H and β_L, and decreases with n.*

1. *Suppose $1 + n > (\beta_H/\beta_L)^{\eta}$ and condition (6) holds. The Bewley–Weil stationary equilibrium coincides with the Becker–Weil stationary equilibrium. Both types of agents own positive amounts of capital: $S^{H*} > S^{L*} > 0$, and $k^* < k_{\beta_L}$.*

2. *Suppose $1 + n \leq (\beta_H/\beta_L)^{\eta}$, or $1 + n > (\beta_H/\beta_L)^{\eta}$ and condition (6) does not hold. In the Bewley–Weil stationary equilibrium, the most patient agents own the entire capital stock and the least patient agents are indebted: $S^{H*} > 0$, $S^{L*} \leq 0$, and $k_{\beta_L} \leq k^* < k_{\beta_H}$.*

Proof. See Appendix A. ■

Proposition 4 implies that a non-degenerate stationary equilibrium is identical in the Bewley–Weil and the Becker–Weil models. When both types of agents accumulate capital, the completeness of capital markets becomes irrelevant.

By contrast, in a Ramsey conjecture stationary equilibrium in the Bewley–Weil model, the least patient agents are in the maximum borrowing state, so their individual, and thus average, savings are negative. The most patient agents hold both the entire capital stock and the debt of the least patient agents. Consequently, the Ramsey conjecture stationary equilibrium in the

Bewley–Weil model is characterized by a lower stationary capital stock and higher between-type inequality than in the Becker–Weil model.[9]

7 Concluding remarks

In this paper, we study a neoclassical growth model with population growth and heterogeneous agents. This model can be interpreted as a Ramsey model with heterogeneous discounting and the entry of new agents with zero initial endowments, or as an OLG model with infinitely-lived agents who differ in their discount factors. We show that there exists a unique stationary equilibrium, which can be one of two types: a Ramsey-conjecture-type, where only the most patient agents own the entire capital stock, or a non-degenerate-type, where less patient agents also hold positive amounts of capital.

The mechanism that invalidates the Ramsey conjecture operates through the OLG demographic structure, where population growth is interpreted as the arrival of new agents who are not included in the economic calculus of existing ones. The entry of new agents without initial endowments raises the interest rate (compared to the Ramsey demographic structure, where existing agents treat new ones as part of their dynasty), thereby incentivizing less patient agents to save. Loosely speaking, population growth mitigates the redistribution of wealth via capital markets from less patient to more patient agents.

We study the impact of fiscal policies on income and inequality in a stationary equilibrium. We show that introducing public debt, a labor income tax, or a capital subsidy shifts the economy from a Ramsey-conjecture-type to a non-degenerate-type equilibrium. These policies reduce intragenerational inequality among agents with different discount factors but increase intergenerational inequality among agents entered in different periods.

Our research can be extended in a number of ways. It is worthwhile to study other assumptions about the relationship between discounting and demography. For instance, different types of agents can have different population growth rates. Moreover, each agent can have an endogenous discount factor depending on their relative income. Overall, this paper provides a tractable framework for analyzing long-run growth and inequality in models with heterogeneous discounting.

[9] This result resembles Becker et al. (2015), who study stationary equilibria under progressively liberalized borrowing. They show that consumption and wealth inequality are lowest in the Ramsey–Becker model (with borrowing constraints) and increase with the number of periods for which borrowing is allowed, reaching a maximum in the Ramsey–Bewley model (with complete capital markets). However, in their case, the stationary capital stock is determined by the discount factor of the most patient agent and is independent of the borrowing regime.

Appendix

A Proof of Theorem 1 and Propositions 3–4

Step 1. Dynamics of the Bewley–Weil model: log-utility. First, we characterize the solution to the Bewley–Weil consumer problem (13). Let

$$W_\tau \;=\; w_\tau + \frac{w_{\tau+1}}{1+r_{\tau+1}} + \frac{w_{\tau+2}}{(1+r_{\tau+1})(1+r_{\tau+2})} + \dots\,,$$

be the present (period-τ) value of the lifetime labor income of an agent entered in period τ. We have

$$W_{\tau+1} \;=\; (1+r_{\tau+1})\,(W_\tau - w_\tau)\;. \tag{A.1}$$

In view of the No-Ponzi Game condition, the sequence of budget constraints in problem (13) can be rewritten as a single intertemporal budget constraint:

$$c_\tau + \frac{c_{\tau+1}}{1+r_{\tau+1}} + \frac{c_{\tau+2}}{(1+r_{\tau+1})(1+r_{\tau+2})} + \dots \;\le\; (1+r_\tau)s^{\tau,i}_{\tau-1} + W_\tau\,. \tag{A.2}$$

Suppose $u(c) = \ln c$. Then the solution to problem (13), $\{c^{\tau,i}_t, s^{\tau,i}_t\}_{t=\tau}^{\infty}$, satisfies the usual first-order conditions $c^{\tau,i}_{t+1} = \beta_i(1+r_{t+1})c^{\tau,i}_t$ for $t \ge \tau$, and is given by

$$c^{\tau,i}_t \;=\; (1-\beta_i)\left((1+r_t)s^{\tau,i}_{t-1} + W_t\right)\,, \qquad t \ge \tau\,, \tag{A.3}$$

$$s^{\tau,i}_t \;=\; (1+r_t)s^{\tau,i}_{t-1} + w_t - c^{\tau,i}_t\,, \qquad t \ge \tau\,. \tag{A.4}$$

Aggregating (A.3) and (A.4) using the fact that $s^{\tau,i}_{\tau-1} = 0$ for $\tau \ge 1$, we have for all $\tau \ge 1$,

$$C^i_\tau \;=\; (1-\beta_i)\left(\frac{1+r_\tau}{1+n}S^i_{\tau-1} + W_\tau\right)\,. \tag{A.5}$$

$$S^i_\tau \;=\; \frac{1+r_\tau}{1+n}S^i_{\tau-1} + w_\tau - C^i_\tau\,. \tag{A.6}$$

Combining (A.1), (A.5) and (A.6), we obtain (5):

$$\begin{aligned} C^i_{\tau+1} \;&=\; (1-\beta_i)(1+r_{\tau+1})\left(\frac{S^i_\tau}{1+n} + W_\tau - w_\tau\right) \\ &=\; (1-\beta_i)(1+r_{\tau+1})\left(\frac{1+r_\tau}{1+n}S^i_{\tau-1} + W_\tau - C^i_\tau - \frac{nS^i_\tau}{1+n}\right) \\ &=\; \beta_i(1+r_{\tau+1})C^i_\tau - (1-\beta_i)(1+r_{\tau+1})\frac{nS^i_\tau}{1+n}\,. \end{aligned}$$

Moreover, if in the Becker–Weil model type i agents make positive savings, then the average consumption level of type i agents also evolves according to (5).

Step 2. Consumer problem under a constant interest rate: isoelastic utility. If $\{(C^{i*}, S^{i*})_{i=H,L}, k^*, r^*, w^*\}$ is a stationary equilibrium in the Becker–Weil model, then each agent entered in period τ solve problem (1) (with non-negativity constraints on savings) given constant sequences of interest rates $\{r^*, r^*, \ldots\}$ and wage rates $\{w^*, w^*, \ldots\}$. We begin by characterizing the solution to corresponding problem (13) in the Bewley–Weil model (without non-negativity constraints on savings). Let

$$W^* \;=\; w^* + \frac{w^*}{1+r^*} + \frac{w^*}{(1+r^*)^2} + \ldots \;=\; (1+r^*)\frac{w^*}{r^*}\,.$$

For isoelastic utility function, the first-order conditions take the form

$$c_{t+1}^{\tau,i*} \;=\; [\beta_i(1+r^*)]^\eta\, c_t^{\tau,i*}\,, \qquad t \geq \tau\,. \tag{A.7}$$

Substituting W^* and $c_{t+1}^{\tau,i*}$ into the intertemporal budget constraint (A.2) which holds as equality, we obtain

$$c_\tau^{\tau,i*}\left(1 + \sum_{t=1}^{\infty} \frac{(\beta_i(1+r^*))^{\eta t}}{(1+r^*)^t}\right) \;=\; \frac{c_\tau^{\tau,i*}}{1-(\beta_i)^\eta(1+r^*)^{\eta-1}} \;=\; (1+r^*)s_{\tau-1}^{\tau,i*} + W^*\,.$$

Because of (A.4) and (A.1), the above equation holds for all periods τ, and hence

$$c_t^{\tau,i*} \;=\; \left[1-(\beta_i)^\eta(1+r^*)^{\eta-1}\right]\left[(1+r^*)s_{t-1}^{\tau,i*} + W^*\right]\,, \qquad t \geq \tau\,. \tag{A.8}$$

Using (A.8) with (A.4), we get the savings dynamics for a type i agent entered in period τ:

$$s_t^{\tau,i*} \;=\; [\beta_i(1+r^*)]^\eta s_{t-1}^{\tau,i*} + \frac{w^*}{r^*}\left([\beta_i(1+r^*)]^\eta - 1\right)\,. \tag{A.9}$$

When $\beta_i(1+r^*) > 1$, $s_t^{\tau,i*} > 0$ even if $s_{\tau-1}^{\tau,i*} = 0$. Hence savings of type i agents in the solution to problem (13) without borrowing constraints are positive for all $\tau > t$, and the solution to problem (1) with borrowing constraints is the same.

When $\beta_i(1+r^*) \leq 1$, for all $\tau \geq 1$, we have $s_\tau^{\tau,i*} \leq 0$. In the Bewley–Weil model, this type i agents borrow against their future wages, and $c_\tau^{\tau,i*}$ is such that they repay this debt in the limit. In the Becker–Weil model, for this type i agents in problem (1) with borrowing constraints we have $s_\tau^{\tau,i*} = 0$ and $c_\tau^{\tau,i*} = w^*$. These agents prefer to consume even more, but cannot save less, since borrowing

is prohibited. By the principle of optimality, for this type i agents the solution to problem (1) is $s_t^{\tau,i*} = 0$ and $c_t^{\tau,i*} = w^*$ for all $t \geq \tau$.

Step 3. Stationary equilibrium characterization. There are three cases. When $1 + r^* \leq 1/\beta_H$, there is no stationary equilibrium in any of the models. When $1/\beta_H < 1 + r^* < 1/\beta_L$, the average consumption dynamics for type H agents are the same in both models, as they make positive savings. For type L agents, in the Becker–Weil model we must have $S^{L*} = 0$. When $1 + r^* > 1/\beta_L$, for both types of agents the non-negativity constraints on savings are not binding, and both types in the both models make positive savings.

Suppose that type i agents make positive savings. Aggregating (A.8) over different generations, and using the fact that $s_{\tau-1}^{\tau,i*} = 0$ for all $\tau \geq 1$, we get

$$C^{i*} \;=\; \left[1 - (\beta_i)^\eta (1+r^*)^{\eta-1}\right] \left(\frac{1+r^*}{1+n} S^{i*} + W^*\right) . \tag{A.10}$$

In a stationary equilibrium, (A.6) becomes

$$C^{i*} = \frac{r^* - n}{1+n} S^{i*} + w^* . \tag{A.11}$$

Combining (A.10) and (A.11), we obtain the stationary version of (5):

$$C^{i*} \;=\; (1+r^*) \left((\beta_i)^\eta (1+r^*)^{\eta-1} C^{i*} - \left[1 - (\beta_i)^\eta (1+r^*)^{\eta-1}\right] \frac{n S^{i*}}{1+n}\right) .$$

Therefore, for this type i we have

$$C^{i*} \;=\; \frac{(1+r^*)\left[1 - (\beta_i)^\eta (1+r^*)^{\eta-1}\right]}{[\beta_i(1+r^*)]^\eta - 1} \; \frac{n S^{i*}}{1+n} . \tag{A.12}$$

Using (A.12) and (A.11), we finally obtain

$$S^{i*} \;=\; (1+n) \; \frac{w^*}{r^*} \; \frac{[\beta_i(1+r^*)]^\eta - 1}{1 + n - [\beta_i(1+r^*)]^\eta} . \tag{A.13}$$

Note that S^{i*} given by (A.13) is positive when $1/\beta_i < 1 + r^* < (1+n)^{1/\eta}/\beta_i$. Otherwise, in a Becker–Weil stationary equilibrium $S^{i*} = 0$. Thus, there exists a stationary equilibrium in which both types of agents own capital only if

$$\frac{1}{\beta_L} \;<\; 1 + r^* \;<\; \frac{(1+n)^{1/\eta}}{\beta_H} . \tag{A.14}$$

In particular, we should have $(1+n)^{1/\eta} > \beta_H/\beta_L$. Note also that since the function $g(\lambda_i^*) = \lambda_i^*/(n-\lambda_i^*)$ increases with λ_i^*, and $\lambda_i^* = [\beta_i(1+r^*)]^\eta - 1$ increases with β_i for all η and r^*, we have $S^{H*} > S^{L*}$.

Introduce the price functions:

$$r(k) \equiv f'(k) - 1, \qquad \text{and} \qquad w(k) \equiv f(k) - kf'(k).$$

In equilibrium, $r^* = r(k^*)$, $w^* = w(k^*)$, and $(1+n)k^* = \pi_H S^{H*} + \pi_L S^{L*}$.

Let $\lambda_i = \lambda_i(r(k)) = [\beta_i(1+r(k))]^\eta - 1$, and

$$H(k) = \frac{w(k)}{k}, \qquad \text{and} \qquad J_i(k) = \pi_i \frac{\lambda_i(r(k))}{n - \lambda_i(r(k))} \frac{1}{r(k)}, \tag{A.15}$$

with the convention that in the Becker–Weil model $J_i(k) \equiv 0$ when $\lambda_i < 0$, that is, for $k > k_{\beta_i}$. Then, by (A.13), the stationary capital stock k^* in both the Becker–Weil and the Bewley–Weil models is a solution to the following equation in k:

$$\Phi(k) \equiv H(k)\ (J_H(k) + J_L(k)) = 1. \tag{A.16}$$

Let $\underline{k} > 0$ be such that $f'(\underline{k}) = (1+n)^{1/\eta}/\beta_H$. It follows from (A.14) that k^* is bounded from below by $\underline{k}$: we have $\underline{k} < k^* < k_{\beta_L}$. Clearly, $J_H(\underline{k}) = \infty$, and $J_H(k_{\beta_H}) + J_L(k_{\beta_H}) \leq 0$ in both models. Then $\Phi(\underline{k}) = \infty$ and $\Phi(k_{\beta_H}) \leq 0$. Therefore, there exists a solution to (A.16), which defines a stationary equilibrium.

The stationary equilibrium is non-degenerate if $S^{L*} > 0$ or $J_L(k^*) > 0$, that is, $k^* \in (\underline{k}, k_{\beta_L})$. Since $J_L(k_{\beta_L}) = 0$, a necessary and sufficient condition for that is $H(k_{\beta_L})J_H(k_{\beta_L}) < 1$, which can be written as

$$\frac{w(k_{\beta_L})}{k_{\beta_L} r(k_{\beta_L})} \frac{(\beta_H)^\eta - (\beta_L)^\eta}{(1+n)(\beta_L)^\eta - (\beta_H)^\eta} < \frac{1}{\pi_H}.$$

Note that

$$\frac{w(k)}{kr(k)} = \frac{f(k)/k - f'(k)}{f'(k) - 1} = \frac{1/\alpha(k) - 1}{1 - 1/f'(k)}, \tag{A.17}$$

where $\alpha(k) = kf'(k)/f(k)$ is the capital income share. Hence there is a solution to Eq. (A.16) within $(\underline{k}, k_{\beta_L})$ if (6) holds. The generalization to I agents with different elasticities η_i is straightforward and yields condition (11). Since all types of agents save, the stationary equilibria in the Bewley–Weil and the Becker–Weil models are the same.

In the Becker–Weil model, the stationary equilibrium is a Ramsey conjecture one if $S^{L*} = 0$, that is, $k^* \in (k_{\beta_L}, k_{\beta_H})$. A necessary and sufficient condition for that is $H(k_{\beta_L})J_H(k_{\beta_L}) \geq 1$, which is the opposite of (6). When there are I agents

with different elasticities η_i, we have to consider the most patient and the second patient agent, which yields condition (12).

In the Bewley–Weil model, both S^{H*} and S^{L*} satisfy (A.13), but now $S^{L*} < 0$. The stationary capital stock k^* is a solution to Eq. (A.16), but in this interval $J_L(k) < 0$. As above, since $J_L(k_{\beta_L}) = 0$, there exists at least one solution to Eq. (A.16) within the interval $(k_{\beta_L}, k_{\beta_H})$ when $H(k_{\beta_L})J_H(k_{\beta_L}) \geq 1$, which is the opposite of (6). Also, $\Phi(k)$ crosses 1 earlier when $J_L(k) < 0$ than when $J_L(k) = 0$, and hence the stationary capital stock in the Becker–Weil model is higher than that in the Bewley–Weil model.

Step 4. Proof of Theorem 1. A stationary equilibrium is unique if the function $H(k)J_i(k)$ monotonically decreases with k for all i. Then there is a unique intersection of $\Phi(k)$ with 1 within $(\underline{k}, k_{\beta_H})$.

Fix type i. Consider the function $J_i(r) = \pi_i\lambda_i(r)/\left[n - \lambda_i(r)\right]/r$. We have

$$\frac{J_i'(r)}{J_i(r)} \;=\; \frac{n\lambda_i'(r)}{\left[n-\lambda_i(r)\right]\lambda_i(r)} - \frac{1}{r} \;=\; \frac{n\eta\beta_i[1+\lambda_i(r)]^{1-1/\eta}}{\left[n-\lambda_i(r)\right]\lambda_i(r)} - \frac{1}{r}\,. \tag{A.18}$$

Furthermore,

$$\frac{H'(k)}{H(k)} \;=\; \frac{w'(k)}{w(k)} - \frac{1}{k} \;=\; \frac{k|r'(k)|}{w(k)} - \frac{1}{k}\,.$$

Denote $\rho = (\sigma-1)/\sigma$, where $0 \leq \rho < 1$, which corresponds to $\sigma \geq 1$. Then the CES production function is $f(k) = A\left(ak^\rho + 1 - a\right)^{1/\rho}$. Let $\varphi = \varphi(k) = ak^\rho + 1 - a$. Then

$$r(k) \;=\; Aak^{\rho-1}\varphi^{(1-\rho)/\rho} - 1\,, \qquad \text{and} \qquad w(k) \;=\; A(1-a)\varphi^{(1-\rho)/\rho}\,.$$

Moreover,

$$\begin{aligned} r'(k) \;&=\; Aa(\rho-1)k^{\rho-2}\varphi^{(1-\rho)/\rho} + Aak^{\rho-1}\left((1-\rho)/\rho\right)\varphi^{(1-2\rho)/\rho}a\rho k^{\rho-1} \;= \\ &\qquad Aa(\rho-1)k^{\rho-2}\varphi^{(1-2\rho)/\rho}\left(\varphi - ak^\rho\right) \;=\; -\frac{(1-\rho)(1-a)(1+r(k))}{k\varphi}\,. \end{aligned}$$

Therefore,

$$\frac{J_i'(k)}{J_i(k)} \;=\; \frac{J_i'(r)r'(k)}{J_i(r)} \;=\; -\frac{(1-\rho)(1-a)}{k\varphi}\left(\frac{\eta n\left[1+\lambda_i(r(k))\right]}{\left[n-\lambda_i(r(k))\right]\lambda_i(r(k))} - \frac{1+r(k)}{r(k)}\right), \tag{A.19}$$

and

$$\frac{H'(k)}{H(k)} \;=\; \frac{k(1-\rho)(1-a)(1+r(k))\varphi^{-1/\rho} - A(1-a)}{A(1-a)k} \;=\; \frac{(1-\rho)ak^{\rho} - \varphi}{k\varphi}\,,$$

so that finally

$$\frac{H'(k)}{H(k)} \;=\; \frac{-\rho ak^{\rho} - (1-a)}{k\varphi} \;<\; 0\,. \tag{A.20}$$

The function $H(k)J_i(k)$ is monotonically decreasing for all k if $H'(k)/H(k) + J_i'(k)/J_i(k) < 0$. By (A.19)–(A.20), this condition takes the form

$$\begin{aligned}\frac{\eta n(1-\rho)(1-a)\left[1+\lambda_i(r(k))\right]}{\left[n-\lambda_i(r(k))\right]\lambda_i(r(k))} \;&>\; (1-\rho)ak^{\rho} - \varphi + (1-\rho)(1-a)\frac{1+r(k)}{r(k)} \\ &=\; \frac{(1-\rho)(1-a)}{r(k)} - \rho\varphi\,.\end{aligned}$$

For $\rho \geq 0$, we have $-\rho\varphi \leq 0$, and it is sufficient to show that for all k such that $\lambda_i(r(k)) \geq 0$,

$$\frac{n\left[1+\lambda_i(r(k))\right]}{\left[n-\lambda_i(r(k))\right]\lambda_i(r(k))} \;>\; \frac{1}{\eta r(k)}\,. \tag{A.21}$$

Since $r(k) = [1+\lambda_i(r(k))]^{1/\eta}/\beta_i - 1$, this condition can be rewritten as

$$\begin{aligned}&\eta n(1+\lambda_i)\left[(1+\lambda_i)^{1/\eta} - \beta_i\right] \;>\; \beta_i(n-\lambda_i)\lambda_i \;\iff \\ &\qquad \eta n(1+\lambda_i)^{1+1/\eta} - \eta n\beta_i(1+\lambda_i) \;>\; \beta_i n\lambda_i - \beta_i\lambda_i^2 \;\iff \\ &\qquad\quad Q(\lambda_i) \;\equiv\; \eta n(1+\lambda_i)^{1+1/\eta} + \beta_i\lambda_i^2 - \beta_i n(1+\eta)\lambda_i - \beta_i n\eta \;>\; 0\,.\end{aligned}$$

Note that $Q(0) = \eta n(1-\beta_i) > 0$. Moreover, $Q'(\lambda_i)$ increases with λ_i, and $Q'(0) = (1+\eta)n(1-\beta_i) > 0$. Thus, $Q(\lambda_i) > 0$ for all $\lambda_i \geq 0$. It follows that $H(k)J_i(k)$ is monotonically decreasing, and hence there exists a unique solution to Eq. (A.16).

For the CES production function, the capital income share is determined as

$$\alpha(k) \;=\; \frac{kf'(k)}{f(k)} \;=\; \frac{Aak^{\rho}[\varphi(k)]^{(1-\rho)/\rho}}{A[\varphi(k)]^{1/\rho}} \;=\; \frac{ak^{\rho}}{\varphi(k)}\,.$$

At the same time, the value k_{β_L} satisfies

$$Aa(k_{\beta_L})^{\rho-1}\left[\varphi(k_{\beta_L})\right]^{(1-\rho)/\rho} \;=\; \frac{1}{\beta_L}\,, \qquad \text{so that} \qquad \frac{(k_{\beta_L})^{\rho}}{\varphi(k_{\beta_L})} \;=\; (Aa\beta_L)^{\rho/(1-\rho)}\,.$$

Therefore, the value of the capital income share in condition (6) is given by $\alpha(k_{\beta_L}) = \left[a\,(A\beta_L)^{\rho}\right]^{1/(1-\rho)}$.

B Proof of Theorem 2

Repeating the argument from Appendix A and aggregating solutions to (8) given constant sequences of interest rates, wage rates and taxes, it is easily seen that the stationary capital stock in the Becker–Weil model with public debt is a solution to

$$\left(H(k) - \frac{\chi f(k)}{k}\left(r(k) - n\right)\right)\left(J_H(k) + J_L(k)\right) - \frac{\chi f(k)}{k} \;=\; 1\,, \tag{B.1}$$

where $H(k)$ and $J_i(k)$ are defined in (A.15). Note that for $\chi = 0$, Eq. (B.1) coincides with Eq. (A.16).

Since $J_H(\underline{k}) = \infty$, and $J_H(k_{\beta_H}) + J_L(k_{\beta_H}) = 0$, there is a solution to Eq. (B.1) within $(\underline{k}, k_{\beta_H})$ whenever $H(\underline{k}) > \chi f(\underline{k})(r(\underline{k}) - n)/\underline{k}$. In other words, there exists an equilibrium in the Becker–Weil model with public debt for all $\chi < \overline{\chi}$, where the maximum debt-to-output ratio is given by

$$\overline{\chi} \;=\; \frac{w(\underline{k})}{f(\underline{k})(r(\underline{k}) - n)} \;=\; \frac{1 - \alpha(\underline{k})}{r(\underline{k}) - n} \;=\; \frac{1 - a^\sigma\left(A\beta_H/(1+n)^{1/\eta}\right)^{\sigma-1}}{(1+n)^{1/\eta}/\beta_H - (1+n)}\,.$$

The uniqueness of a solution to (B.1) requires monotonicity of the left-hand side, which is not easily analytically tractable. However, since the left-hand side of (B.1) decreases with χ, any solution to (B.1) is lower than the (unique) solution to the corresponding equation for $\chi = 0$.

By the same argument as in Appendix A, there exists at least one non-degenerate equilibrium in the Becker–Weil model with public debt when the value of the left-hand side of (B.1) at k_{β_L} is lower than one. When condition (6) holds, this is true for any χ. When condition (6) does not hold, this requirement holds for all $\chi > \chi^*$, where the threshold debt-to-output ratio χ^* satisfies

$$\left(\frac{1/\alpha(k_{\beta_L}) - 1}{1 - \beta_L} - \frac{\chi^*}{\beta_L \alpha(k_{\beta_L})}\frac{1 - \beta_L(1+n)}{1 - \beta_L}\right)\frac{\pi_H[(\beta_H)^\eta - (\beta_L)^\eta]}{(1+n)(\beta_L)^\eta - (\beta_H)^\eta} - \frac{\chi^*}{\beta_L \alpha(k_{\beta_L})} \;=\; 1\,.$$

C Proof of Theorem 3

Repeating the argument from Appendix A and taking into account that the after-tax stationary interest rate is $(1-\theta_K)(1+r^*)$, it is easily seen that there are three cases. When $1 + r^* \le 1/[(1-\theta_K)\beta_H]$, there is no stationary equilibrium. When $1/[(1-\theta_K)\beta_H] < 1 + r^* < 1/[(1-\theta_K)\beta_L]$, type H agents save, while $S^{L*} = 0$. When $1 + r^* > 1/[(1-\theta_K)\beta_L]$, both types of agents make positive savings.

Focus on the non-degenerate case, as the Ramsey conjecture case can be considered similarly. Let $\tilde{r}(k) = (1 - \theta_K)r(k) - \theta_K$, that is,

$1+\tilde{r}(k)=(1-\theta_K)(1+r(k))$. Aggregating solutions to individual problems with taxes, given constant sequences of interest rates and wage rates, it is straightforward to verify that the stationary capital stock in the Becker–Weil model with income taxation is a solution to

$$(1-\theta_N)\frac{w(k)}{k}\left(\pi_H\frac{\lambda_H(\tilde{r}(k))}{n-\lambda_H(\tilde{r}(k))}\;\frac{1}{\tilde{r}(k)}+\pi_L\frac{\lambda_L(\tilde{r}(k))}{n-\lambda_L(\tilde{r}(k))}\;\frac{1}{\tilde{r}(k)}\right)\;=\;1\,,\qquad\text{(C.1)}$$

or $(1-\theta_N)H(k)\left(\tilde{J}_H(k)+\tilde{J}_L(k)\right)=1$, where $\lambda_i(\tilde{r}(k))=[\beta_i(1+\tilde{r}(k))]^\eta-1$, and, as in (A.15),

$$\tilde{J}_i(k)\;=\;\tilde{J}_i(\tilde{r}(k))\;=\;\pi_i\;\frac{\lambda_i(\tilde{r}(k))}{n-\lambda_i(\tilde{r}(k))}\;\frac{1}{\tilde{r}(k)}\,.$$

Repeating the derivation of (A.19) from Appendix A, we obtain

$$\frac{\tilde{J}_i'(k)}{\tilde{J}_i(k)}\;=\;-\frac{(1-\rho)(1-a)}{k\varphi}\left(\frac{\eta n\left[1+\lambda_i(\tilde{r}(k))\right]}{\left[n-\lambda_i(\tilde{r}(k))\right]\lambda_i(\tilde{r}(k))}-\frac{1+\tilde{r}(k)}{\tilde{r}(k)}\right)\,.$$

It is easily seen that (A.21) also holds for $\tilde{r}(k)$ instead of $r(k)$. Therefore, for $0\le\rho<1$, the function $(1-\theta_N)H(k)\tilde{J}_i(k)$ is monotonically decreasing for all k. For any given pair $(\theta_K,\theta_N)\in[0,1)\times[0,1)$, Eq. (C.1) has a unique solution, and there is a unique stationary equilibrium with income taxation.

Clearly, the left-hand side of (C.1) decreases with θ_N for any k. Hence the stationary capital stock k^* decreases with θ_N. Furthermore, the left-hand side of (C.1) decreases with θ_K if each $J_i(\tilde{r})=\pi_i\lambda_i(\tilde{r})/\left[n-\lambda_i(\tilde{r})\right]/\tilde{r}$ increases with $\tilde{r}$. It follows from (A.18) that $J_i'(\tilde{r})>0$ if and only if

$$([\beta_i(1+\tilde{r})]^\eta-1)^2+n\left(1+\eta\beta_i^\eta(1+\tilde{r})^{\eta-1}\tilde{r}-[\beta_i(1+\tilde{r})]^\eta\right)\;>\;0\,.$$

The sufficient condition for the above inequality is $1+\eta\beta_i^\eta(1+\tilde{r})^{\eta-1}\tilde{r}\ge[\beta_i(1+\tilde{r})]^\eta$, which holds for all $\eta\ge1$, because the function $(1+\tilde{r})^\eta$ is convex. Thus, the stationary capital stock k^* decreases with θ_K for $\eta\ge1$.

Moreover, by the implicit function theorem, it follows from (C.1) that

$$\frac{dk^*}{d\theta_K}\;=\;\frac{(1+r)\sum_i J_i'(\tilde{r})/\sum_i J_i(\tilde{r})}{H'/H+(1-\theta_K)r'\sum_i J_i'(\tilde{r})/\sum_i J_i(\tilde{r})}\,,$$

and it is easily seen that when $J_i'(\tilde{r})>0$, we have $[dk^*/d\theta_K](1-\theta_K)r'<1+r$. Therefore, $d\tilde{r}/d\theta_K=[dk^*/d\theta_K](1-\theta_K)r'-(1+r)<0$ for $\eta\ge1$.

Let $\underline{\tilde{k}}$ be such that $f'(\underline{\tilde{k}})=(1+n)^{1/\eta}/[(1-\theta_K)\beta_H]$. A unique stationary equilibrium is non-degenerate if a solution to Eq. (C.1) lies within $(\underline{\tilde{k}},k_{(1-\theta_K)\beta_L})$. Since $\tilde{J}_L(k_{(1-\theta_K)\beta_L})=0$, we need $(1-\theta_N)H(k_{(1-\theta_K)\beta_L})\tilde{J}_H(k_{(1-\theta_K)\beta_L})<1$.

Since for a given θ_K, the left-hand side decreases with θ_N, there exists a unique non-degenerate stationary equilibrium for any pair (θ_K, θ_N) with $\theta_N \geq \theta_N^*(\theta_K)$, where the function $\theta_N^*(\theta_K)$ is implicitly defined by the equation $(1-\theta_N)H(k_{(1-\theta_K)\beta_L})\tilde{J}_H(k_{(1-\theta_K)\beta_L}) = 1$, or

$$(1-\theta_N)\frac{[1/\alpha(k_{\beta_L})][1/(1-\theta_K)^{1/(1-\rho)}] - 1/(1-\theta_K)}{1-\beta_L}\,\frac{(\beta_H)^\eta - (\beta_L)^\eta}{(1+n)(\beta_L)^\eta - (\beta_H)^\eta} \;=\; \frac{1}{\pi_H}\,,$$

where we have used the fact that for the CES production function,

$$\alpha(k_{(1-\theta_K)\beta_L}) \;=\; \frac{k_{(1-\theta_K)\beta_L} f'(k_{(1-\theta_K)\beta_L})}{f(k_{(1-\theta_K)\beta_L})} \;=\; \alpha(k_{\beta_L})\,(1-\theta_K)^{\rho/(1-\rho)}\,. \qquad \text{(C.2)}$$

Clearly, $\lim_{\theta_K \to 1} \theta_N^*(\theta_K) = 1$. Furthermore,

$$\theta_N^*(0) \;=\; 1 - \frac{1}{\pi_H}\,\frac{(1+n)(\beta_L)^\eta - (\beta_H)^\eta}{(\beta_H)^\eta - (\beta_L)^\eta}\,\frac{1-\beta_L}{1/\alpha(k_{\beta_L}) - 1}\,,$$

and $\theta_N^*(0) \geq 0$ whenever condition (6) does not hold, that is, whenever the model without taxes admits a Ramsey conjecture equilibrium.

Finally, let us show that $\theta_N^*(\theta_K)$ is concave. Recall that $\sigma = 1/(1-\rho)$. Clearly, for $\sigma = 1$, $\theta_N^*(\theta_K)$ is linear. Let $1/(1-\theta_N) \equiv y$ and $1/(1-\theta_K) \equiv x$. Also, let for brevity $\alpha(k_{\beta_L}) \equiv \alpha$. Then the definition of $\theta_N^*(\theta_K)$ takes the form

$$(1-\theta_N^*(0))\,(1-\alpha)\,y \;=\; x^\sigma - \alpha x\,.$$

Ignoring the constant multipliers, by the implicit function theorem, we have

$$\frac{d\theta_N^*}{d\theta_K} \;=\; \frac{1}{y}\,\frac{\sigma x^{\sigma+1} - \alpha x^2}{x^\sigma - \alpha x} \;=\; \frac{\sigma x^{\sigma-1} - \alpha}{(x^{\sigma-1} - \alpha)^2}\,.$$

For any $\sigma > 1$, the right-hand side decreases with x. Indeed, the sign of its derivative coincides with the sign of $2 - \sigma - \sigma x^{\sigma-1}/\alpha$, which is negative, since $\sigma x^{\sigma-1}/\alpha + \sigma > 2$, as $\alpha < 1$ and $x \geq 1$. Thus, $\theta_N^*(\theta_K)$ is strictly concave for $\sigma > 1$.

D Proof of Theorem 4

Focus on the non-degenerate case, as the Ramsey conjecture case can be considered similarly. Let $1 + \hat{r}(k) = (1-\vartheta)(1+r(k))$ and $\lambda_i(\hat{r}(k)) = [\beta_i(1+\hat{r}(k))]^\eta - 1$. Repeating the argument from Appendix A, the stationary capital stock under redistributive taxation is a solution to the equation $H(k)\left(\hat{J}_H(k) + \hat{J}_L(k)\right) = 1$,

where

$$\hat{J}_i(k) \;=\; \pi_i \,\frac{\lambda_i(\hat{r}(k))}{[n-\lambda_i(\hat{r}(k))]\,r(k) - n\vartheta(1+r(k))}\,. \tag{D.1}$$

Let $\underline{\hat{k}}$ be such that $\underline{\hat{r}} = f'(\underline{\hat{k}}) - 1$ is a solution to the following equation in r:

$$(1+n-[\beta_H(1-\vartheta)(1+r)]^{\eta}) \;=\; n\vartheta\frac{1+r}{r}\,.$$

Note that this solution exists for any capital subsidy $\vartheta < 0$, and for sufficiently small capital tax $0 \leq \vartheta < \overline{\vartheta}$. Since $\hat{J}_H(\underline{\hat{k}}) = \infty$, and $\hat{J}_H(k_{(1-\vartheta)\beta_H}) + \hat{J}_L(k_{(1-\vartheta)\beta_H}) = 0$, there exists k^* within $(\underline{\hat{k}}, k_{(1-\vartheta)\beta_H})$ if ϑ is such that $\underline{\hat{k}} < k_{(1-\vartheta)\beta_H}$, that is, $\beta_H(1-\vartheta)(1+\underline{\hat{r}}) > 1$. Hence there exists an equilibrium under redistributive taxation for all $\vartheta \in (\underline{\vartheta}, \overline{\vartheta})$, where the maximum capital subsidy is $\underline{\vartheta} = -\underline{\hat{r}}/\,(1+\underline{\hat{r}})$.

Differentiating (D.1) with respect to k, we obtain

$$\frac{\hat{J}_i'(k)}{\hat{J}_i(k)} \;=\; r'(k)\frac{[\lambda_i(\hat{r}(k))]^2 - n(1-\vartheta)\,[\lambda_i(\hat{r}(k))] + \eta n\,[1+\lambda_i(\hat{r}(k))]\,[r(k)/(1+r(k)) - \vartheta]}{([n-\lambda_i(\hat{r}(k))]\,r(k) - n\vartheta(1+r(k)))\,\lambda_i(\hat{r}(k))}\,.$$

There exists a unique stationary equilibrium under redistributive taxation if $H'(k)/H(k) + \hat{J}_i'(k)/\hat{J}_i(k) < 0$. For $0 \leq \rho < 1$, this condition holds if

$$\begin{aligned}[\lambda_i(\hat{r}(k))]^2\,[1+r(k)] - n(1-\vartheta)\,[\lambda_i(\hat{r}(k))]\,[1+r(k)] + \eta n\,[1+\lambda_i(\hat{r}(k))]\,\hat{r}(k) \\ >\;\; (n\vartheta(1+r(k)) + [\lambda_i(\hat{r}(k)) - n]\,r(k))\,\lambda_i(\hat{r}(k))\,,\end{aligned}$$

which follows from (A.21) for $\hat{r}(k)$. Thus, $H(k)\hat{J}_i(k)$ is monotonically decreasing, and for any given $\vartheta \in (\underline{\vartheta}, \overline{\vartheta})$, there exists a unique k^*.

A unique stationary equilibrium is non-degenerate if $k^* < k_{(1-\vartheta)\beta_L}$. Since $\hat{J}_L(k_{(1-\vartheta)\beta_L}) = 0$, a necessary and sufficient condition is $H(k_{(1-\vartheta)\beta_L})\hat{J}_H(k_{(1-\vartheta)\beta_L}) < 1$, which, taking into account (A.17) and (C.2), can be written as

$$\frac{\left(1/\alpha(k_{\beta_L})/\,(1-\vartheta)^{\rho/(1-\rho)} - 1\right)[(\beta_H)^{\eta} - (\beta_L)^{\eta}]}{[(1+n)(\beta_L)^{\eta} - (\beta_H)^{\eta}]\,[1-\beta_L(1-\vartheta)] - n\vartheta(\beta_L)^{\eta}} \;<\; \frac{1}{\pi_H}\,. \tag{D.2}$$

It is easily checked that the left-hand side is increasing in ϑ. Therefore, there exists a non-degenerate stationary equilibrium for all $\vartheta < \vartheta^*$, where the threshold value ϑ^* satisfies (D.2) with equality. If the model without taxes admits a Ramsey conjecture equilibrium, then condition (6) does not hold, and hence the left-hand side of (D.2) at $\vartheta = 0$ is higher than $1/\pi_H$, implying that $\vartheta^* < 0$.

The stationary capital stock decreases with ϑ if each $\hat{J}_i$ decreases with ϑ.

Differentiating (D.1) with respect to θ, we get (ignoring the positive denominator)

$$\frac{d\hat{J}_i}{d\vartheta} \;=\; -n(1+r)\left(1+\eta\beta_i^{\eta}(1+\hat{r})^{\eta-1}\hat{r}-[\beta_i(1+\hat{r})]^{\eta}\right)\,,$$

which is negative when $\eta \geq 1$, because the function $(1+\hat{r})^{\eta}$ is convex.

Moreover, differentiating (D.1) with respect to r, we obtain (again, ignoring the same denominator)

$$\frac{d\hat{J}_i}{dr} \;=\; 1+[\beta_i(1+\hat{r})]^{2\eta}+n(1-\vartheta)\left(1+\eta\beta_i^{\eta}(1+\hat{r})^{\eta-1}\hat{r}-[\beta_i(1+\hat{r})]^{\eta}\right)\,.$$

Repeating the argument from Appendix C, it is easily checked that the after-tax interest rate decreases with ϑ, that is, $d\hat{r}/d\vartheta = [dk^*/d\vartheta](1-\vartheta)r' - (1+r) < 0$, if

$$\left|\frac{d\hat{J}_i}{d\vartheta}\right|\,(1-\vartheta) \;\leq\; \frac{d\hat{J}_i}{dr}\,(1+r)\,,$$

which is true for any ϑ whenever $\eta \geq 1$.

E Proof of Proposition 1

In a Ramsey conjecture stationary equilibrium, $S^{L*} = 0$, and hence the Gini index for $k_{\beta_L} < k^* < k_{\beta_H}$ is $G^* = 1 - \pi_H$. In a non-degenerate stationary equilibrium, S^{i*} is given by (A.13), and hence, by (9), the Gini index for $\underline{k} < k^* < k_{\beta_L}$ is

$$G^* \;=\; \frac{\pi_L J_H(k^*) - \pi_H J_L(k^*)}{J_H(k^*) + J_L(k^*)}\,.$$

We prove that G^* increases with k^* for $\underline{k} < k^* < k_{\beta_L}$. Note that G^* increases with the ratio $J_H(k^*)/J_L(k^*)$. Using (A.19), for $\sigma \geq 1$ ($0 \leq \rho < 1$), we have

$$\begin{aligned}\left(\frac{J_H(k)}{J_L(k)}\right)'\,\frac{J_L(k)}{J_H(k)} \;&=\; \frac{J_H'(k)}{J_H(k)} - \frac{J_L'(k)}{J_L(k)} \\ &= \frac{\eta n(1-\rho)(1-a)}{k\varphi}\,\frac{(1+n)\left[\lambda_H(r(k)) - \lambda_L(r(k))\right]}{\left[n-\lambda_L(r(k))\right]\lambda_L(r(k))\left[n-\lambda_H(r(k))\right]\lambda_H(r(k))} \;>\; 0\,,\end{aligned}$$

since for each k, $\lambda_H(r(k)) > \lambda_L(r(k))$.

Let $\overline{n}$ be such that (6) holds as equality:

$$\frac{(\beta_H)^{\eta} - (\beta_L)^{\eta}}{(1+\overline{n})(\beta_L)^{\eta} - (\beta_H)^{\eta}}\,\frac{1/\alpha(k_{\beta_L}) - 1}{1-\beta_L} \;=\; \frac{1}{\pi_H}\,.$$

Then for all $n > \overline{n}$, the unique stationary equilibrium is non-degenerate. We prove that G^* decreases with n for $n > \overline{n}$. Indeed, it is easily seen that for each given k^*, the ratio $J_H(k^*)/J_L(k^*)$ decreases with n. Thus, the direct effect of an increase in n is negative. By Theorem 1, k^* decreases with n, and since G^* increases with k^*, the indirect effect is negative as well.

F Proof of Proposition 2

Suppose that $S^{i*} > 0$, that is, type i agents make positive savings. In a stationary equilibrium, for a type i agent entered in period τ, the dynamics of savings is given by (A.9). Therefore, for agents entered in period $\tau \geq 1$ with zero initial endowments, the wealth level in period t is given by

$$s_t^{\tau,i*} \;=\; \lambda_i^* \frac{w^*}{r^*}\left[1 + (1+\lambda_i^*) + \ldots + (1+\lambda_i^*)^t\right] \;=\; \frac{w^*}{r^*}\left[(1+\lambda_i^*)^{t+1-\tau} - 1\right],$$

where $1 + \lambda_i^* = [\beta_i(1+r^*)]^\eta$. Similarly, for agents from the very first generation initially present in period 0, the wealth level in period t is given by

$$s_t^{0,i*} \;=\; \frac{w^*}{r^*}\left[(1+\lambda_i^*)^{t+1} - 1\right] + (1+\lambda_i^*)^{t+1}\frac{S^{i*}}{1+n}\,.$$

Hence $s_\tau^{0,i*} > s_\tau^{1,i*} > \ldots > s_\tau^{\tau,i*}$ for each $\tau \geq 1$.

We prove that for any agent type i, for any two generations N and M with $0 \leq N < M \leq \tau$, the ratio $s_\tau^{N,i*}/s_\tau^{M,i*}$ increases with λ_i^* for all τ. If $N > 0$, then this ratio has the form

$$\frac{s_\tau^{N,i*}}{s_\tau^{M,i*}} \;=\; \frac{(1+\lambda_i^*)^{\tau+1-N} - 1}{(1+\lambda_i^*)^{\tau+1-M} - 1}\,.$$

Since $N < M$, it is easily checked that the right-hand side increases with λ_i^*.

If $N = 0$, then, using (A.13), the ratio $s_\tau^{0,i*}/s_\tau^{M,i*}$ takes the form

$$\frac{(1+\lambda_i^*)^{\tau+1}\left([S^{i*}/(1+n)]\,[r^*/w^*] + 1\right) - 1}{(1+\lambda_i^*)^{\tau+1-M} - 1} \;=\; \frac{(1+\lambda_i^*)^{\tau+1}\,[n/(n-\lambda_i^*)] - 1}{(1+\lambda_i^*)^{\tau+1-M} - 1}\,.$$

Since $n/(n-\lambda_i^*) > 1$ and $n/(n-\lambda_i^*)$ increases with λ_i^*, the numerator grows faster than $(1+\lambda_i^*)^{\tau+1} - 1$, and hence the ratio $s_\tau^{0,i*}/s_\tau^{M,i*}$ also increases with λ_i^*.

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References

Aguiar, M., Bils, M., and Boar, C. (2025). Who are the Hand-to-Mouth? *Review of Economic Studies*, **92** (3), 1293–1340.

Becker, R. A. (1980). On the Long-run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households. *Quarterly Journal of Economics*, **95**, 375–382.

Becker, R. A. (2006). Equilibrium Dynamics with Many Agents. In Mitra, T., Dana, R.-A., Le Van, C., and Nishimura, K., editors, *Handbook on Optimal Growth 1: Discrete Time*, 385–442. Springer, Berlin, Heidelberg.

Becker, R. A., Borissov, K., and Dubey, R. S. (2015). Ramsey Equilibrium with Liberal Borrowing. *Journal of Mathematical Economics*, **61**, 296–304.

Becker, R. A. and Mitra, T. (2012). Efficient Ramsey Equilibria. *Macroeconomic Dynamics*, **16** (S1), 18–32.

Becker, R. A. and Zilcha, I. (1997). Stationary Ramsey Equilibria under Uncertainty. *Journal of Economic Theory*, **75** (1), 122–140.

Bewley, T. F. (1982). An Integration of Equilibrium Theory and Turnpike Theory. *Journal of Mathematical Economics*, **10**, 233–267.

Blanchard, O. J. (1985). Debt, Deficits, and Finite Horizons. *Journal of Political Economy*, **93** (2), 223–247.

Borissov, K. and Pakhnin, M. (2023). Economic Growth Models with Heterogeneous Discounting. *Computational Mathematics and Mathematical Physics*, **63** (3), 337–359.

Bosi, S. and Seegmuller, T. (2010). On the Role of Progressive Taxation in a Ramsey Model with Heterogeneous Households. *Journal of Mathematical Economics*, **46** (6), 977–996.

Buiter, W. H. (1988). Death, Birth, Productivity Growth and Debt Neutrality. *Economic Journal*, **98** (391), 279–293.

Caballé, J. (1998). Growth Effects of Taxation under Altruism and Low Elasticity of Intertemporal Substitution. *Economic Journal*, **108** (446), 92–104.

Drugeon, J.-P. and Wigniolle, B. (2017). On Impatience, Temptation and Ramsey's Conjecture. *Economic Theory*, **63** (1), 73–98.

Epstein, L. G. and Hynes, J. A. (1983). The Rate of Time Preference and Dynamic Economic Analysis. *Journal of Political Economy*, **91** (4), 611–635.

Espino, E. (2005). On Ramsey's Conjecture: Efficient Allocations in the Neoclassical Growth Model with Private Information. *Journal of Economic Theory*, **121** (2), 192–213.

Li, W. and Sarte, P. D. (2004). Progressive Taxation and Long-run Growth. *American Economic Review*, **94** (5), 1705–1716.

Lucas, R. E. and Stokey, N. L. (1984). Optimal Growth with Many Consumers. *Journal of Economic Theory*, **32** (1), 139–171.

Mankiw, N. G. (2000). The Savers-Spenders Theory of Fiscal Policy. *American Economic Review*, **90**, 120–125.

Michel, P. and Pestieau, P. (2005). Fiscal Policy with Agents Differing in Altruism and Ability. *Economica*, **72**, 121–135.

Mitra, T. and Sorger, G. (2013). On Ramsey's Conjecture. *Journal of Economic Theory*, **148** (5), 1953–1976.

Pichler, P. and Sorger, G. (2009). Wealth Distribution and Aggregate Time-preference: Markov-perfect Equilibria in a Ramsey Economy. *Journal of Economic Dynamics and Control*, **33** (1), 1–14.

Ramsey, F. P. (1928). A Mathematical Theory of Saving. *Economic Journal*, **38** (152), 543–559.

Saez, E. (2013). Optimal Progressive Capital Income Taxes in the Infinite Horizon Model. *Journal of Public Economics*, **97**, 61–74.

Sarte, P. D. (1997). Progressive Taxation and Income Inequality in Dynamic Competitive Equilibrium. *Journal of Public Economics*, **66** (1), 145–171.

Sorger, G. (2002). On the Long-run Distribution of Capital in the Ramsey Model. *Journal of Economic Theory*, **105** (1), 226–243.

Sorger, G. (2008). Strategic Saving Decisions in the Infinite-horizon Model. *Economic Theory*, **36** (3), 353–377.

Weil, P. (1989). Overlapping Families of Infinitely-lived Agents. *Journal of Public Economics*, **38** (2), 183–198.

Weil, P. (2008). Overlapping Generations: The First Jubilee. *Journal of Economic Perspectives*, **22** (4), 115–134.